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Spinorial Conformal Blocks: Dirac Action and Integrability

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Abstract

It has previously been established ([IS16]) that the Casimir equation, that defines conformal blocks, coincides for a particular choice of coordinates with the time-independent Schrödinger equation of the BC_2-Calogero–Sutherland model, which is an integrable model whose integrability and solutions have been studied using Dunkl operators and the representation theory of particular Hecke algebras.

To further explore this relation between conformal blocks and the Calogero–Sutherland model, we develop a general formalism to have not only invariant differential operators (like the quadratic Casimir element) but also invariant Clifford algebra-valued differential operators (like an appropriate Dirac operator) act on solutions to the conformal Ward identities. We then compute the action of Kostant's cubic Dirac operator for the (almost) scalar case and obtain a matrix whose entries can be expressed using Dunkl operators.

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Introduction

Why CFT?

Symmetries are one of the cornerstones of modern theoretical physics. In quantum field theory (QFT), a central role is being played by the "symmetry group of spacetime", i.e. the group of isometries (metric-preserving diffeomorphisms) of the pseudo-Riemannian manifold we want to define QFTs on. Invariance under this group is one of the fundamental requirements for a QFT. This is why every QFT textbook (e.g. [Sre07]) and set of lecture notes (e.g. [Ton]) under the sun includes – directly or indirectly – some discussion of the Poincaré and Lorentz groups and their representation theory.

If we choose to enlarge this symmetry group to the group of conformal transformations, i.e. angle-preserving diffeomorphisms, diffeomorphisms that are allowed to scale the metric, we are in the realm of *conformal field theory* (CFT). Invariance under this larger symmetry group severely constrains a theory, to the point where we can associate with it a "small" set of parameters that fully determines it: the *CFT data*.

We are only going to consider CFT on flat spacetime (be it Minkowski or Euclidean) here. In that case it turns out that CFT in two dimensions is qualitatively (and quantitatively) very different from > 2 dimensions. We are only going to talk about $d > 2$ here.

In this setting (and also for $d = 2$ flat spacetime), one class of conformal transformations is dilation, i.e. expansion or contraction. Conformal invariance then implies that the physical phenomena that a CFT describes are scale-invariant. This is not the whole story, as there are scale-invariant QFTs that are not conformal [FGS11], but it is a big part of it.

Now, it might seem that a scale-invariant QFT is not of much use. After all, the real world is very much scale-dependent, and for example the presence of massive particles in our theory would introduce an energy scale and therefore violate scale-invariance. So how physically relevant can a scale-invariant QFT (and by extension, a CFT) really be? Roughly speaking, there are two fields of application: high-energy physics and statistical mechanics. The former uses Lorentzian signature, and the latter Euclidean.

Universality Classes

The first point of relevance is on its face a purely theoretical one. Regardless of conformal symmetry, we can always investigate the effect of dilation on a QFT. This will usually change the theory (e.g. by changing parameters), so in effect it induces a (smooth) dynamical system on "theory space": the renormalisation group flow (RG flow). A dynamical system that we can study like any other (smooth) dynamical system: by studying its fixed points [1], i.e. by studying scale-invariant QFTs.

Any orbit of the RG flow could then theoretically be periodic or connect two (not necessarily different) fixed points. As the former does not happen in practice, we can associate to every QFT the two endpoints of its RG flow orbit: the *UV-* and *IR-limits* for maximum contraction and maximum dilation, respectively. This means that we can classify QFTs by the properties of their UV- and IR-limits. In particular:

- Are they CFTs?
- If yes, which ones?

Under this classification, we say that QFTs fall into *universality classes*.

Critical Phenomena

A more concrete example where we can observe scale-invariant QFTs in real life is that of critical opalescence. As is outlined e.g. in [BRW19, section 5.5], critical opalescence is a consequence of there being density fluctuations of all scales at the critical point, or to paraphrase one of the authors directly: "not just bubbles a few millimetres in diameter, but bubbles of all sizes, bubbles inside other bubbles, bubbles intersecting with each other" (January 2020, as part of the university course *Statistical Physics and Condensed Matter Theory, extension*). This suggests that the theory governing the behaviour of density fluctuations at the critical point is scale-invariant.

This turns out to be correct, and moreover, for any condensed matter system that has a critical point, the behaviour at that critical point is governed by its IR-limit, according to [Zin96, section 25]. This means that the aforementioned universality classes not only describe some abstract high- and low-energy limits of QFTs, but actually describe critical behaviour, such as critical exponents of certain quantities, of real-world (and non-real-world) condensed matter systems. Further references include [Hen99].

Cosmology and Quantum Gravity

Kicked off in 1998 by [Mal98], a hypothesised correspondence between (supersymmetric) type-IIB string theory on anti-de Sitter backgrounds and (super) CFTs on the boundaries of these backgrounds has become an important field of study within cosmology and string theory. This is known as the AdS/CFT-correspondence, which has become an important tool in tackling quantum gravity problems such as the black hole information paradox (see [Haw05]). An also slightly dated but very extensive review can be found at (all the papers referenced in) [Bei+12], and for an introduction, see [Năs15].

More generally, the AdS/CFT correspondence is the prime example of the *holographic principle*, the notion that quantum gravity in some region (bulk) of spacetime can be described with a theory on the (lower-dimensional) boundary of that region. In other

[1] as away from the fixed points, every flow can be "rectified" and hence can be transformed to look "the same". This is the rectification theorem, e.g. [Arn78, section 7.1]

words: that like a hologram, the bulk is fully described by the boundary theory. This principle was first established by [Sus95].

$d = 2$ CFT in String Theory

A more hands-on relevance for string theory is provided by the fact that path integral computations involve fields on the 2-dimensional world sheet that are conformally invariant. Examples include the *bc* Faddeev–Popov ghost system, see e.g. [Pol05, section 3.3]. This turns out, however, to have little in common with the subject of this thesis, as 2-dimensional CFTs have a much larger symmetry algebra: the (infinite-dimensional) Virasoro algebra, see [Sch08, chapter 5]. This constrains 2-dimensional CFTs even more, and gives rise to sophisticated techniques like vertex operator algebras and the use of complex analysis, which are inapplicable to the $d > 2$ case we're going to be concerned with.

Conformal Bootstrap

Now we've seen some of the sources of CFTs and some of the reasons why they are a relevant topic, and remembered that each CFT is fully determined by a "small" set of parameters: the CFT data. A natural next step is now to work the other way around, and ask which CFT data could conceivably correspond to a CFT. This is achieved by using conformal invariance and in particular the operator product expansion (Section 1.4) to derive properties of the n-point correlations function: the conformal Ward identities (Section 1.2) and the crossing equations (Section 1.5). In the spirit of Wightman's Reconstruction theorem ([Sch08, theorem 8.18]) and the Osterwalder–Schrader theorem ([Sch08, section 8.6]), correlation functions that satisfy these axioms then give rise to a CFT.

The quest for "reconstructing" CFTs thus becomes the task of solving the crossing equations. It turns out to be sufficient to solve the crossing equation for the 4-point function. Now, this is still a functional equation, which in general is hard to solve. To help with that, it proves useful to expand the 4-point function as a series in "nice" functions, called *conformal blocks* (Section 1.6), that satisfy the conformal Ward identities and that are eigenfunctions of the quadratic Casimir element of the conformal algebra. If we substitute this expansion into the crossing equation, we obtain system of quadratic equations for the coefficients (see (1.7)).

By applying linear functionals to both sides, we obtain equations (of numbers!) that can be solved. The art of choosing the correct (i.e. most useful) functional involves a lot of knowledge about the conformal blocks and their properties, and a lot of analytical and numerical (e.g. [Sim15]) techniques have been developed to aid with that.

This procedure is known as the *conformal bootstrap* and was first started by [Pol74] and [FGG73]. Reviews of the current state of the art can be found at [PRV19], [Har+22], and [PS22].

Integrable Systems

Another reason why symmetries are so relevant to modern physics is Noether's theorem, which states that every (smooth) symmetry produces a conserved quantity or current. Let us now discuss the role that conserved quantities are going to play in this thesis. For that we abandon quantum mechanics for the moment and turn our heads to Hamiltonian mechanics.

Assume that our phase space M has dimension $2n$, then a Hamiltonian function H is called *integrable* if there are n commuting linearly independent conserved quantities, i.e. linearly independent functions $f_i : M \to \mathbb{R}$ satisfying

$$\{H, f_i\} = 0 \quad (i = 1, \dots, n), \qquad \{f_i, f_j\} = 0 \quad (i, j = 1, \dots, n)$$

($\{\cdot, \cdot\}$ are the Poisson brackets). Under mild conditions (the level sets of H have to be compact), there exists a canonical transformation to *action-angle coordinates* $(w_1, \dots, w_n, J_1, \dots, J_n)$ where the transformed Hamiltonian $\tilde{H}$ does not depend on the generalised positions $w_1, \dots, w_n$. As a consequence, time evolution is given by

$$\dot{w}_i = \frac{\partial \tilde{H}}{\partial J_i} = \nu_i(J_1, \dots, J_n), \qquad \dot{J}_i = -\frac{\partial \tilde{H}}{\partial w_i} = 0,$$

where ν_i only depends on $J_1, \dots, J_n$ and is hence time-independent. Furthermore, the coordinates $J_1, \dots, J_n$ are determined by the values of the conserved quantities $f_1, \dots, f_n$. To summarise the property of integrability in a sentence: "There are enough conserved quantities to uniquely determine time evolution of any point in phase space."

Example: Two-Body System with Central Force

Consider $M = \mathbb{R}^{12}$, describing two bodies in $\mathbb{R}^3$, one having position $x = (x_1, x_2, x_3)$ and momentum p, the other having position X and momentum P, let $V : \mathbb{R} \to \mathbb{R}$ be strictly monotonically increasing and smooth. Define

$$H(x, X, p, P) := \frac{\|p\|^2}{2m} + \frac{\|P\|^2}{2M} + V(\|x - X\|).$$

This system is evidently translation and rotation symmetric. By Noether's theorem, this implies the conservation of total momentum $p + P$ and total angular momentum $x \times p + X \times P$. These are six independent conserved quantities, hence this Hamiltonian is integrable. The extra condition of the existence of action-angle coordinates is also satisfied.

Example: Calogero–Sutherland Model

Let $A = \mathbb{R}^N_{>0}$, $\mathfrak{a} = \mathbb{R}^N$ and let $R \subseteq \mathfrak{a}^*$ be a root system with $R^+ \subseteq R$ a set of positive roots. Let $k : R \to \mathbb{R}$ be a real-valued label that is invariant under the action of R's Weyl group W.

Define $H : A \times \mathfrak{a}^* \to \mathbb{R}$

$$H(\exp(u), \lambda) := \frac{1}{2}\|\lambda\|^2 + \sum_{\alpha \in R^+} \frac{k_\alpha(k_\alpha + 2k_{2\alpha} - 1)\|\alpha\|^2}{8\sinh^2\left(\frac{\alpha(u)}{2}\right)}.$$

This is the *Calogero–Sutherland (CS) model*, which was shown in [OP76][2] to be integrable.

For the case where R is of type A_n we obtain

$$H(\exp(u), \lambda) = \frac{\|\lambda\|^2}{2} + \sum_{i>j} \frac{k(k-1)}{2\sinh^2\left(\frac{u_i - u_j}{2}\right)},$$

a system of n particles in one dimension that interact via a csch^2-potential. And, as a slightly more involved example, when R is of type BC_n, we obtain

$$\begin{aligned}
H(\exp(u), \lambda) &= \frac{\|\lambda\|^2}{2} + \sum_{i=1}^n \left(\frac{k_1(k_1 + 2k_3 - 1)}{4\sinh^2(\frac{u_i}{2})} + \frac{4k_3(k_3-1)}{4\sinh^2(u_i)} \right) \\
&\quad + \sum_{i>j} \left(\frac{k_2(k_2-1)}{2\sinh^2\left(\frac{u_i - u_j}{2}\right)} + \frac{k_2(k_2-1)}{2\sinh^2\left(\frac{u_i + u_j}{2}\right)} \right) \\
&=: \frac{\|\lambda\|^2}{2} + \sum_{i=1}^n V_{PT}(u_i) \\
&\quad + \frac{1}{4}k_2(k_2-1)\sum_{i \neq j}(V(u_i - u_j) + V(u_i + u_j)) \\
&\quad + k_3(k_3-1)\sum_{i=1}^n V(2u_i)
\end{aligned}$$

So we have again an interaction by a csch^2-potential, but in addition, our particles are also subject to an "external" potential V_{PT} [3], and the interaction also involves $u_i + u_j$-dependency. We can interpret this as interaction as follows: the particle with coordinate $\exp(u_i)$ has an image particle at $\exp(-u_i)$, and the particles interact not only with each other but also with each other's images. In particular, the interaction between a particle and its own image may have a different interaction energy ($4k_3(k_3 - 1)$ instead of $k_2(k_2 - 1)$).

If we canonically quantise this Hamiltonian system (and *then* obtain what is usually referred to as the CS model), we obtain a quantum integrable system that admits solutions in terms of of Heckman–Opdam hypergeometric functions (details in [Opd00, section 6] and [HS94, chapters 2 and 4]). Both the CS model and the Heckman–Opdam hypergeometric functions have been extensively studied with the use of special differential

[2]only for the classical root systems, but it's also true for the exceptional ones

[3]"PT" stands for Pöschl–Teller, the authors of the paper [PT33] that first considered csch^2-potentials in quantum mechanics

reflection operators called *Dunkl(–Cherednik) operators* and the representation theory of certain degenerate affine Hecke algebras. We will see more of that in the second part of this thesis.

Relation to Conformal Blocks

Having now introduced both CFT and integrable models, we should also address the question why they are relevant to the same master thesis. The answer to this question was provided six years ago by [IS16] with more in-depth explanation two years later in [IS18], in which is shown that the Casimir eigenvalue equation for conformal blocks (for four scalar fields) is actually conjugate to the Schrödinger equation of the BC_2-CS model, so that we can use all the theory about Dunkl operators to address scalar conformal blocks. Since then, various efforts, e.g. [SSI17], [SS18], have been undertaken to generalise this correspondence to "spinning" conformal blocks, i.e. conformal blocks corresponding to non-scalar fields and to a matrix version of the CS model. Further generalisations include a generalisation to defect blocks (i.e. operators supported on defects) [Isa+18], and to n-point functions with $n > 4$ [Bur+21]. However, a general theory remains elusive at time of writing.

In [IS16] it is pointed out that this correspondence between the Casimir eigenvalue equation and the CS model can be understood as follows: the (action of the) quadratic Casimir element is the radial part of the Laplace–Beltrami operator on the conformal group G with respect to a subgroup K (i.e. the Laplace–Beltrami operator acting on functions that satisfy a left and right equivariance condition with respect to K). These radial parts have been shown e.g. by [HS94, proposition 5.1.5] (in a slightly different setting) to be related to a sum of squares of Dunkl operators and thereby the CS model.

The correspondence "Laplace–Beltrami operator" $\leftrightarrow$ "sum of squares of Dunkl operators" begs the question that if we take a different invariant operator on the left side, could we maybe obtain single Dunkl operators on the right side? Now, since Dunkl operators are of order 1, any invariant differential operator we apply on the left would have to have order 1 as well, and there usually (in semisimple Lie algebras) any invariant 1st-order differential operators. However, if we scout around, slightly widen our horizons, we are quickly reminded of a question that led to relativistic quantum mechanics: is there a square root of the Laplacian? And indeed, if we consider the answer to that question, the Dirac operator, we quickly notice that it is an invariant differential operator of order 1 that is matrix-valued (or *Clifford algebra*-valued to be precise).

This leads us to some of the central questions we're going to consider in this thesis: what is the general theory of having Clifford algebra-valued invariant differential operators act on (spinning) conformal blocks, and is there a relation between the action of an appropriate Dirac operator and the CS model?

To put it in physics language: from [IS16] we know that scalar conformal blocks are solutions to a (quantum) integrable system that is obtained by having the Laplacian act on the conformal blocks. If instead we apply its "square root", the Dirac operator, to a spinorial conformal block, can we recover traces of our integrable system that allow us to achieve a deeper understanding of this conformal block $\leftrightarrow$ CS model correspondence?

Outline

Let's now see how this thesis is structured. We are first going to dive a bit deeper into CFTs (Section 1.1) and make precise our introduction section (Sections 1.2, 1.4) as well as the line of thought that leads towards conformal blocks (Section 1.6) and the crossing equation (Section 1.5). Then, after briefly introducing some useful tools (Chapter 2) that are going to be used throughout this thesis, we will analyse the structure of the conformal group (Chapter 3) and the structure of the space of point configurations (Section 3.3). Then, we will introduce induced representations (Chapter 4), make more concrete some statements about conformal blocks, and then recap/develop the general theory of invariant (Clifford algebra-valued and scalar) differential operators on conformal blocks (Sections 4.2 and 4.3, respectively). This leads us to considering the actions of quadratic Casimir element and (Kostant's cubic) Dirac operator on scalar/as-scalar-as-possible[4] conformal blocks (Chapter 5).

Afterwards we will introduce some of the general theory of Dunkl operators (Section 6.3), the Hecke algebras (Section 6.4), their representation theory (Section 6.5), and work towards expressing the differential operators from before in terms of Dunkl operators (Sections 6.7 and 6.8, respectively).

[4]Any action of a Dirac operator involves a representation of a Clifford algebra. We will take the smallest such representation

Contents

1. Setup

Let $p + q = d > 2$ be natural numbers with $p \geq q$. Most commonly we will encounter $q = 0$ or $q = 1$. In the following we will refer to the vector space $\mathbb{R}^{p,q} = \mathbb{R}^d$, equipped with the standard bilinear form η of signature (p, q), as (p, q)-*spacetime*.

1.1. Conformal Fields

For ordinary quantum field theory, the Wightman axioms state that we can interpret quantum field operators, elements of a set we're calling $\mathscr{F}\mathscr{O}$, to be tempered operator-valued distributions on spacetime, that carry a representation of the Poincaré group $SO(p, q) \ltimes \mathbb{R}^{p,q}$ (cf. Appendix A.1 for the precise meaning of these words). And they are usually classified by means of this representation. For conformal field theories, we are enhancing the symmetry group to the group $G := SO(p + 1, q + 1)_0$, whose Lie algebra can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$, with

$$\begin{aligned}
\mathfrak{m} &= \operatorname{span}\{F_{\mu\nu} \mid 1 \leq \mu, \nu \leq d\} \cong \mathfrak{so}(p, q) \\
\mathfrak{a} &= \mathbb{R}D \cong \mathbb{R} \\
\mathfrak{n} &= \operatorname{span}\{K^\mu \mid 1 \leq \mu \leq d\} \\
\overline{\mathfrak{n}} &= \operatorname{span}\{P^\mu \mid 1 \leq \mu \leq d\}.
\end{aligned}$$

These basis elements satisfy the commutation relations

$$\begin{aligned}
[F^{\mu\nu}, F^{\rho\sigma}] &= \eta^{\nu\rho} F^{\mu\sigma} + \eta^{\mu\sigma} F^{\nu\rho} - \eta^{\nu\sigma} F^{\mu\rho} - \eta^{\mu\rho} F^{\nu\sigma} \\
[F^{\mu\nu}, K^\rho] &= \eta^{\nu\rho} K^\mu - \eta^{\mu\rho} K^\nu \\
[F^{\mu\nu}, P^\rho] &= \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu \\
[D, K^\mu] &= K^\mu \\
[D, P^\mu] &= -P^\mu \\
[K^\mu, P^\nu] &= 2F^{\mu\nu} + 2\eta^{\mu\nu} D
\end{aligned}$$

(with all other commutators being zero). We can see that $\mathfrak{m} \oplus \overline{\mathfrak{n}}$ is the Poincaré algebra from "ordinary" quantum field theory, D is taken to represent dilations, and the K^μ are *special conformal transformations* (SCTs). It is also handy to introduce $\mathfrak{q} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, which is a Lie subalgebra. Analogously, we can introduce the subgroups

$$M \cong SO(p, q), \qquad A \cong \mathbb{R}, \qquad N, \overline{N} \cong \mathbb{R}^d$$

associated to these subalgebras, and their product $Q := MAN \leq G$, which is also a group, a maximal parabolic subgroup, as we're going to see in Section 3.2.

In particular, for $q = 0$ we have

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, m \in SO(p) \right\}$$

and for $q = 1$ we have

$$M = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| \, m \in SO(p,1) \right\}$$

where ± 1 is chosen depending on if m lies in the identity component or not.

We can then choose our fields $(\psi_i)_{i \in I}$ such that

$$\begin{aligned} (F_{\mu\nu} \cdot \psi_i)(0) &= \sum_{j \in I} \pi(F_{\mu\nu})^j{}_i \psi_j(0) \\ (D \cdot \psi_i)(0) &= \Delta_i \psi_i(0) \\ (K^\mu \cdot \psi_i)(0) &= 0 \end{aligned} \tag{1.1}$$

for $\Delta_i \in \mathbb{R}$ and (π, V) a representation of $SO(p,q)$. In that case we say that the $(\psi_i)_{i \in I}$ are *primary fields*, with scaling dimensions $(\Delta_i)_{i \in I}$ that transform as the representation π. We can reconstruct the entire representation of G on our fields ψ_i from this data: as part of the Poincaré algebra, the operators P_μ represent translations, and hence we have

$$(\exp(-x^\mu P_\mu) \cdot \psi_i)(0) = \psi_i(x).$$

Note that here and in the remainder of this thesis, we're going to abuse notation and write distributions as generalised functions.

To get the actions of $\mathfrak{m}, \mathfrak{a}, \mathfrak{n}$ away from 0, we can leverage the commutators of these subalgebras with $\bar{\mathfrak{n}}$ and obtain

$$\begin{aligned} (F_{\mu\nu} \cdot \psi_i)(x) &= \pi(F_{\mu\nu})^j{}_i \psi_j(x) + x_\nu \partial_\mu \psi_i(x) - x_\mu \partial_\nu \psi_i(x) \\ (D \cdot \psi_i)(x) &= \Delta_i \psi_i(x) - x^\mu \partial_\mu \psi_i(x) \\ (K_\mu \cdot \psi_i)(x) &= -2x^\nu \Sigma_{\mu\nu}{}^j{}_i \psi_j(x) - 2x_\mu(\Delta_i - x^\nu \partial_\nu)\psi_i(x) - x^2 \partial_\mu \psi_i(x). \end{aligned} \tag{1.2}$$

Thus we've described a representation of $\mathfrak{g}$. Usually we have a bit more: a representation of the Lie group G. In order to attempt to describe that representation, we will make some notational simplifications. Recall that V was what we called the representation space of π, and let $(v^i)_{i \in I}$ be the basis of V that gives rise to the matrix elements $\pi(F_{\mu\nu})^j{}_i$ used earlier:

$$F_{\mu\nu} \cdot v^i = \pi(F_{\mu\nu}) v^i = \pi(F_{\mu\nu})^i{}_j v^j.$$

Without loss of generality, we can choose the v^i to be compatible with a given decomposition of V into irreps, so that there is a partition $(I_\alpha)_{\alpha \in J}$ so that $(v^i)_{i \in I_\alpha}$ is the basis of an irreducible component of V.

We can now define the tensor operator-valued distribution (tensor field operator)

$$\psi : \mathcal{S}(\mathbb{R}^d) \to V \otimes \mathscr{O}(\mathcal{H}), \psi := v^i \psi_i,$$

where $\mathscr{O}(\mathcal{H})$ denotes the densely defined linear operators on $\mathcal{H}$. Then

$$\begin{aligned}(F_{\mu\nu} \cdot \psi)(0) &= v^i \otimes (F_{\mu\nu} \cdot \psi_i)(0) \\ &= v^i \otimes \pi(F_{\mu\nu})^j{}_i \psi_j \\ &= \pi(F_{\mu\nu})^j{}_i v^i \otimes \psi_j \\ &= (\pi(F_{\mu\nu}) v^i) \otimes \psi_i \\ &= \pi(F_{\mu\nu})\psi(0).\end{aligned}$$

Next, extend the representation π to $\mathfrak{q}$ by having $\pi(D)v^i = \Delta_i v^i$ and $\pi(K_\mu) = 0$. Then the Equations 1.1 reduce to

$$(\xi \cdot \psi)(0) = \pi(\xi)\psi(0) \qquad \xi \in \mathfrak{q}.$$

Since π lifts to a representation of $Q \cong (SO(p, q) \times \mathbb{R}) \ltimes \mathbb{R}^{p,q}$ [1], we can use this to define the generalised function

$$\Psi : \overline{N}Q \text{ “}\to\text{” } V \otimes \mathscr{O}(\mathcal{H}), (\exp(x^\mu P_\mu)p) \mapsto \pi(p^{-1})\psi(x).$$

Now, the set $\overline{N}Q$ is dense in G, and since we assume that the representation from (1.2) lifts to a group representation, we can continue Ψ to all of G. The left regular representation of G,

$$(g \cdot \Psi)(h) := \Psi(g^{-1}h),$$

is then the aforementioned lift.

For a distributionally more sound definition, write Δ_Q for Q's modular function (note that we can pick Haar measures such that $\mu_G = \mu_{\overline{N}} \otimes (\Delta_Q \cdot \mu_Q)$). Then define

$$\Psi : C_c^\infty(G) \to V \otimes \mathscr{O}(\mathcal{H})$$

as

$$\int_Q \Delta_Q(q)\pi(q^{-1})\psi(i(R_q(f))) \,\mathrm{d}\mu_Q\,(q)$$

where $i(f)(x) := f(\exp(x^\mu P_\mu))$ ($x \in \mathbb{R}^{p,q}$) and where $(R_g(f))(h) = f(hg)$. The integral is to be understood weakly: for $v \in D$ and $w \in \mathcal{H}$ we define

$$\langle w, \Psi(f)v\rangle := \int_Q \Delta_Q(q)\pi(q^{-1})\langle w, \psi(i(R_q(f)))v\rangle \,\mathrm{d}\mu_Q\,(q)$$

This integral is finite because the set

$$\{q \in Q \mid i(R_q(f)) \neq 0\} = \left\{q \in Q \;\middle|\; \exists \overline{n} \in \overline{N} : \overline{n}q \in \mathrm{supp}(f)\right\},$$

[1] it started out as a representation of $SO(p, q)$, and the analytic subgroups for $\mathfrak{a}, \mathfrak{n}$ are simply connected

the image of the relatively compact set $\overline{N}Q \cap \operatorname{supp}(f)$ under the continuous projection map $\overline{N}Q \to Q$, is also compact. Ψ then satisfies

$$\begin{aligned}\Psi(R_p(f)) &= \int_Q \Delta_Q(q)\pi(q^{-1})\psi(i(R_{qp}(f)))\,\mathrm{d}\mu_Q\,(q)\\ &= \int_Q \Delta_Q(q)\pi(p)\pi(qp)^{-1}\psi(i(R_{qp}(f)))\,\mathrm{d}\mu_Q\,(q)\\ &= \pi(p)\int_Q \Delta_Q(q)\pi(q)^{-1}\psi(i(R_q(f)))\,\mathrm{d}\mu_Q\,(q)\\ &= \pi(p)\Psi(f),\end{aligned}$$

where we used that $\Delta_Q \cdot \mu_Q$ is a right Haar measure.

1.2. Correlation Functions

The axioms for (conformal) quantum field theory further require the existence of a notion of an expectation value $\langle\cdot\rangle$ that can be taken of the fields, and which is invariant under the operation of shifting all fields by $g \in G$. Using our simplifications from earlier, we can define vector-valued distributions

$$\begin{aligned}&g_n : (\mathbb{R}^{p,q})^n \to V^{\otimes n},\\ &(x_1, \ldots, x_n) \mapsto \langle \psi(x_1) \otimes \cdots \otimes \psi(x_n) = \langle \psi_{i_1}(x_1) \cdots \psi_{i_n}(x_n)\rangle v^{i_1} \otimes \cdots \otimes v^{i_n}\end{aligned}$$

(see Appendix A.1 for tensor products of tensor field operators) and

$$G_n : G^n \to V^{\otimes n}, (x_1, \ldots, x_n) \mapsto \langle \Psi(x_1) \otimes \cdots \otimes \Psi(x_n)\rangle$$

(where any elements of $V^{\otimes n}$ are just pulled out of $\langle\cdot\rangle$). These are called the *n-point functions* or *correlators*.

The distributions G_n then satisfy

$$G_n(x_1p_1, \ldots, x_np_n) = \pi(p_1^{-1}) \otimes \cdots \otimes \pi(p_n^{-1})G_n(x_1, \ldots, x_n) \tag{1.3}$$

for $p_1, \ldots, p_n \in Q$ and

$$G_n(gx_1, \ldots, gx_n) = G_n(x_1, \ldots, x_n). \tag{1.4}$$

These equations (or rather, what they correspond to for g_n), are called the *conformal Ward identities*.

These transformation properties show that we're dealing with

$$(\operatorname{Ind}_Q^G(\pi)^{\otimes n})^G \cong (\operatorname{Ind}_{Q^n}^{G^n}(\pi^{\otimes n}))^{\Delta^n G}$$

i.e. the invariants of a Kronecker product of induced representations, either in the noncompact picture (g_n) or in the induced picture (G_n). ($\Delta^n G$ is the diagonal subgroup $\{(g, \ldots, g) \in G^n \mid g \in G\}$.)

That we're working with $G = SO(p,q)$, dictates that our fields have boson statistics, i.e. commute when taking the expectation value. We therefore expect the following behaviour of our expectation values:

$$\langle \psi_{i_1}(x_1) \cdots \psi_{i_n}(x_n) \rangle = \langle \psi_{i_{\sigma(1)}}(x_{\sigma(1)}) \cdots \psi_{i_{\sigma(n)}}(x_{\sigma(n)}) \rangle \tag{1.5}$$

for $\sigma \in S_n$. If we let S_n act on $V^{\otimes n}$ by permuting tensor factors, and call this representation Σ, this equation reads

$$G_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \Sigma(\sigma)^{-1} G_n(x_1, \ldots, x_n).$$

Note that to simplify calculations, we can decompose V into irreducible Q-modules

$$V = \bigoplus_{\alpha \in J} V_\alpha$$

(by requiring the N-part to be trivial and the A-part to be one-dimensional, we can ensure that V is semisimple), so that

$$\mathrm{Ind}_Q^G(\pi)^{\otimes n} \cong \bigoplus_{\alpha_1, \ldots, \alpha_n \in J} \mathrm{Ind}_Q^G(\pi_{\alpha_1}) \otimes \cdots \otimes \mathrm{Ind}_Q^G(\pi_{\alpha_n})$$

(J indexes the irreducible components of the Q-module (V, π)), so it makes sense to consider the more general (seeming) case where we're inducing from (potentially different) irreducible representations $\pi_1, \ldots, \pi_n$.

We will see later that the covariance conditions are already enough to completely fix the 1-, 2-, and 3-point functions. For the case of π_i trivial on M and with scaling dimension Δ_i this follows as follows:

Example 1.2.1 (1-Point Function)**.** $G_1(g) = G_1(1)$ *is a constant.*

Example 1.2.2 (n-Point Function)**.** *Due to translational symmetry we have*

$$g_n(x_1, \ldots, x_n) = g_n(0, x_2 - x_1, \ldots, x_n - x_1) =: h_n(x_2 - x_1, \ldots, x_n - x_1).$$

Due to rotational symmetry we have

$$h_n(mv_1, \ldots, mv_{n-1}) = h_n(v_1, \ldots, v_{n-1})$$

for all $m \in SO(p,q)$, showing that h_n only depends on the scalars $\eta(v_i, v_j)$ ($i, j = 1, \ldots, n-1$). Besides, we have

$$\eta(v_i, v_j) = \eta(x_{i+1} - x_1, x_{j+1} - x_1) = \frac{1}{2}(x_{i+1,j+1}^2 - x_{1,i+1}^2 - x_{1,j+1}^2),$$

showing that there is k_n such that

$$g_n(x_1, \ldots, x_n) = k_n(x_{12}^2, \ldots, x_{1n}^2, x_{23}^2, \ldots, x_{n-1,n}^2).$$

Here, k_n takes $\frac{n(n-1)}{2}$-many scalar inputs. Next, scaling invariance shows that

$$\begin{aligned}g_n(ax_1,\dots,ax_n) &= k_n(a^2x_{12}^2,\dots,a^2x_{1n}^2,a^2x_{23}^2,\dots,a^2x_{n-1,n}^2)\\ &= a^{-\Delta_1-\dots-\Delta_n}g_n(x_1,\dots,x_n)\\ &= a^{-\Delta_1-\dots-\Delta_n}k_n(x_{12}^2,\dots,x_{1n}^2,x_{23}^2,\dots,x_{n-1,n}^2),\end{aligned}$$

hence k_n is homogeneous of degree $\frac{-\Delta_1-\dots-\Delta_n}{2}$.

Lastly, using Proposition B.4.2, inversion invariance shows that

$$g_n(x_1,\dots,x_n) = \|x_1\|^{-2\Delta_1}\cdots\|x_n\|^{-2\Delta_n}g_n\left(-\frac{w_0wx_1}{\|x_1\|^2},\dots,-\frac{w_0wx_n}{\|x_n\|^2}\right). \tag{1.6}$$

Note that

$$\begin{aligned}\left\|-\frac{w_0wx_i}{\|x_i\|^2}+\frac{w_0wx_j}{\|x_j\|^2}\right\|^2 &= \left\|\frac{x_i}{\|x_i\|^2}-\frac{x_j}{\|x_j\|^2}\right\|^2\\ &= \frac{1}{\|x_i\|^2}+\frac{1}{\|x_j\|^2}-\frac{2\langle x_i,x_j\rangle}{\|x_i\|^2\|x_j\|^2}\\ &= \frac{1}{\|x_i\|^2}+\frac{1}{\|x_j\|^2}+\frac{x_{ij}^2-\|x_i\|^2-\|x_j\|^2}{\|x_i\|^2\|x_j\|^2}\\ &= \frac{x_{ij}^2}{\|x_i\|^2\|x_j\|^2},\end{aligned}$$

hence, (1.6) *becomes*

$$k_n((x_{ij}^2)_{i<j}) = \|x_1\|^{-2\Delta_1}\cdots\|x_n\|^{-2\Delta_n}k_n\left(\left(\frac{x_{ij}^2}{\|x_i\|^2\|x_j\|^2}\right)_{i<j}\right).$$

Together with scaling invariance, this fixes the dependence on n variables, so that ultimately k_n depends (relatively freely) only on

$$\frac{n(n-1)}{2}-n = \frac{n(n-3)}{2}$$

variables. For $n=2$, this equals -1, so the function is even overdetermined and is forced to be zero in some cases. For $n=3$, this equals 0, so the 3-point function is fixed up to a constant, and for $n=4$, this equals 2, which we will see more in-depth later.

Example 1.2.3 (2p Function)**.** *For the 2-point function, the inversion behaviour shows that*

$$\begin{aligned}k_2(x_{12}^2) &= \|x_1\|^{-2\Delta_1}\|x_2\|^{-2\Delta_2}k_2\left(\frac{x_{12}^2}{\|x_1\|^2\|x_2\|^2}\right)\\ &= \left(\frac{\|x_2\|}{\|x_1\|}\right)^{\Delta_{12}}k_2(x_{12}^2),\end{aligned}$$

so either $k_2=0$ or $\Delta_1=\Delta_2$.

Example 1.2.4 (3p Function). *For the 3-point function, the inversion behaviour shows that*

$$k_3(x_{23}^2, x_{31}^2, x_{12}^2) = \frac{k_3\Big(\frac{x_{23}^2}{\|x_2\|^2\|x_3\|^2}, \frac{x_{31}^2}{\|x_3\|^2\|x_1\|^2}, \frac{x_{12}^2}{\|x_1\|^2\|x_2\|^2}\Big)}{\|x_1\|^{2\Delta_1}\|x_2\|^{2\Delta_2}\|x_3\|^{2\Delta_3}}$$
$$= \frac{k_3(x_1^2 x_{23}^2, x_2^2 x_{31}^2, x_3^2 x_{12}^2)}{\|x_1\|^{\Delta_1+\Delta_2+\Delta_3}\|x_2\|^{\Delta_1-\Delta_2+\Delta_3}\|x_3\|^{\Delta_1+\Delta_2-\Delta_3}},$$

which shows that

$$k_3(a^2u, b^2v, c^2w) = \frac{1}{a^{-\Delta_1+\Delta_2+\Delta_3}b^{\Delta_1-\Delta_2+\Delta_3}c^{\Delta_1+\Delta_2-\Delta_3}}k_3(u,v,w),$$

i.e. that

$$g_3(x_1,x_2,x_3) = \frac{k_3(1,1,1)}{x_{23}^{-\Delta_1+\Delta_2+\Delta_3}x_{31}^{\Delta_1-\Delta_2+\Delta_3}x_{12}^{\Delta_1+\Delta_2-\Delta_3}}.$$

1.3. Aside: Fermion Statistics

Instead of looking at $G = SO(p,q)_0$, we could also consider $\mathrm{Spin}(p,q)_0$, which allows for non-integer-spin representations, and therefore for fermionic fields. These then anticommute with each other, but still commute with the bosonic fields.

The way we can formalise this is that we require a decomposition $V = V^+ \oplus V^-$ into even and odd subspaces that is compatible with the $\mathfrak{g}$-isotypic decomposition. We then define the map $\tau \in \mathrm{End}(V \otimes V)$ by

$$\tau((v^+ + v^-) \otimes (w^+ + w^-)) := w^+ \otimes v^+ + w^- \otimes v^+ + w^+ \otimes v^- - w^- \otimes v^-,$$

which flips its inputs and negates the odd elements that were exchanged with odd elements. Now it turns out that τ satisfies the braid relations, and we can build a representation Σ of S_n on $V^{\otimes n}$ by having

$$(i, i+1) \mapsto \mathrm{id}_V^{\otimes i-1} \otimes \tau \otimes \mathrm{id}_V^{\otimes n-i-1},$$

i.e. by acting on the i-th and $i+1$-st tensor factor with τ. If G_n is an n-point function, the permutation behaviour of (1.5) generalises to

$$G_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \Sigma(\sigma)^{-1} G_n(x_1, \ldots, x_n)$$

1.4. Operator-Product Expansions (OPEs)

An important feature of CFTs is that products of field operators evaluated at different points can be developed around one of the points:

$$\psi_i(x)\psi_j(y) = \sum_{k \in I} \lambda_{ijk} f_{ijk}(x,y,z)\psi_k(z)|_{z=y},$$

where equality is to be seen in terms of asymptotic expansions in $x-y$, see Appendix A.2. Here, $f_{ijk}(x, y, z)$ is an asymptotic series (in $x - y$) of differential operators in z.

If we define $f(x, y, z)$ to be a $\mathrm{Hom}(V, V \otimes V)$-valued differential operator so that

$$f(x, y, z)v^k = v^i \otimes v^j f_{ijk}(x, y, z),$$

then the desired intertwining property reads

$$f(x, y, z)(\xi \cdot \psi)(z) = (1 \otimes \xi + \xi \otimes 1)f(x, y, z)\psi(z),$$

where $U(\mathfrak{g}) \otimes 1$ acts on x and the $V \otimes 1$, and $1 \otimes U(\mathfrak{g})$ acts on y and $1 \otimes V$.

Note that since composition of operators is associative (ignoring potential differences in domain), the same should hold for the OPE.

Now, this equality as asymptotic expansion doesn't provide any information about convergence, so even calling it equality is a stretch. However, it turns out, when applied to the vacuum state, or more generally within a correlation function, the OPE actually has a nonempty region of convergence (for details, see [PRV19, section II] or [Mac77])[2]. This means that n-point functions can be reduced to the severely restricted two-point functions, and thus be calculated. Thus, all correlation functions (and hence the theory) are fixed by the representation π, the scaling dimensions Δ_i, and the coefficients λ_{ijk}. These are collectively known as the *CFT data*.

1.5. Crossing Equations

We can now flip this process on its head and ask which CFT data produces a valid CFT. In addition to unitarity constraints and causality/reflection positivity, an important requirement is the aforementioned associativity of the OPE. In particular, if we reduce

$$\langle \psi_{i_1}(x_1) \cdots \psi_{i_n}(x_n) \rangle$$

using the OPE, it can't matter in which order we do this. That is, we want

$$\begin{aligned} &\sum_{k \in I} \lambda_{i_1 i_2 k} f_{i_1 i_2 k}(x_1, x_2, y) \langle \psi_k(y) \psi_{i_3}(x_3) \cdots \psi_{i_n}(x_n) \rangle |_{y=x_2} \\ =&\sum_{k \in I} \lambda_{i_2 i_3 k} f_{i_2 i_3 k}(x_2, x_3, y) \langle \psi_{i_1}(x_1) \psi_k(y) \psi_{i_4}(x_4) \cdots \psi_{i_n}(x_n) \rangle |_{y=x_3} \end{aligned}$$

(or similarly for any other pair of operators next to each other). This equation is called the *crossing equation*. We have

$$\begin{aligned} \langle \psi_{i_1}(x_1) \cdots \psi_{i_4}(x_4) \rangle &= \sum_{k \in I} \lambda_{i_1 i_2 k} f_{i_1 i_2 k}(x_1, x_2, y) \langle \psi_k(y) \psi_{i_3}(x_3) \psi_{i_4}(x_4) \rangle |_{y=x_2} \\ &= \sum_{k, k' \in I} \lambda_{i_1 i_2 i_k} \lambda_{k' i_3 i_4} f_{i_1 i_2 k}(x_1, x_2, y) f_{i_3 i_4 k'}(x_3, x_4, y') \cdot \\ &\qquad \langle \psi_k(y) \psi_{k'}(y') \rangle |_{y=x_2, y'=x_4}. \end{aligned}$$

[2]for a more distributional point of view, cf. [KQR20] and [KQR21]

Since the 2-point function is only nonzero if $v^k, v^{k'}$ live in irreps that are dual to each other, this sum becomes a single sum, say

$$= \sum_{k\in I} \lambda_{i_1 i_2 k}\lambda_{i_3 i_4 \tilde{k}} f_{i_1 i_2 k}(x_1, x_2, y) f_{i_3 i_4 \tilde{k}}(x_3, x_4, y')\langle \psi_k(y)\psi_{\tilde{k}}(y')\rangle|_{y=x_2, y'=x_4}$$
$$=: \sum_{k\in I} \lambda_{i_1 i_2 k}\lambda_{i_3 i_4 \tilde{k}} W^k_{i_1 i_2 i_3 i_4}(x_1, \ldots, x_4).$$

The functions W^k are called conformal partial waves (CPWs) or conformal blocks (though the literature, e.g. [PRV19], reserves the name "conformal blocks" for slightly modified CPWs).

To reduce our number of indices, define

$$W^\alpha(x_1, \ldots, x_4) := \sum_{k\in I_\alpha} v^{i_1} \otimes \cdots \otimes v^{i_4} W^k_{i_1\cdots i_4}(x_1, \ldots, x_4)$$

where $I_\alpha \subseteq I$ is such that $(v^i)_{i\in I_\alpha}$ is a basis for the irreducible component V_α of V. The map

$$\Pi_\alpha : V \to V_\alpha \subseteq V, a_i v^i \mapsto \sum_{i\in I_\alpha} a_i v^i,$$

i.e. the projection onto V_α, is a Q-intertwiner, hence it gives us an intertwiner of induced representations. In other words

$$(g \cdot (\Pi_\alpha \psi)) = \Pi_\alpha(g \cdot \psi)$$

for $g \in G$, and similarly for differential operators. Our CPW becomes

$$W^\alpha(x_1, \ldots, x_4) = (f(x_1, x_2, y)\Pi_\alpha) \otimes (f(x_3, x_4, y')\Pi_{\tilde{\alpha}})\langle \psi(y) \otimes \psi(y')\rangle|_{y=x_2, y'=x_4}.$$

In this index-free way, the four-point function can be written as

$$g_4(x_1, \ldots, x_4) = \sum_{\alpha\in J} \lambda^\alpha \otimes \lambda^{\tilde{\alpha}} W^\alpha(x_1, \ldots, x_4),$$

where $\lambda^\alpha(v^i \otimes v^j) = \lambda_{ijk} v^i \otimes v^j$ (no summation), where $k \in J_\alpha$. That f already acts as an intertwiner, dictates that the λ_{ijk} are constant on irreducibles.

Using the CPWs, the 4-point crossing equation becomes

$$\begin{aligned}\sum_{k\in I} \lambda_{i_1 i_2 k}\lambda_{i_3 i_4 \tilde{k}} W^k_{i_1, \ldots, i_4}(x_1, \ldots, x_4) &= \langle \psi_{i_1}(x_1) \cdots \psi_{i_4}(x_4)\rangle \\ &= \pm\langle \psi_{i_3}(x_3)\psi_{i_2}(x_2)\psi_{i_1}(x_1)\psi_{i_4}(x_4)\rangle \\ &= \pm \sum_{k\in I} \lambda_{i_2 i_3 k}\lambda_{i_1 i_4 \tilde{k}} W^k_{i_3 i_2 i_1 i_4}(x_3, x_2, x_1, x_4),\end{aligned} \tag{1.7}$$

where $\pm$ occurs according to whether or not switching ψ_{i_1} and ψ_{i_3} incurs a sign or not. Index-free, this reads as

$$\sum_{\alpha\in J} \lambda^\alpha \otimes \lambda^{\tilde{\alpha}} W^\alpha(x_1, \ldots, x_4) = \sum_{\alpha\in J} \Sigma((13))\lambda^\alpha \otimes \lambda^{\tilde{\alpha}} W^\alpha(x_3, x_2, x_1, x_4).$$

1.6. Conformal Blocks

For $\alpha \in J$ we have

$$\begin{aligned}((g,g,g,g)\cdot W^\alpha)(x_1,\dots,x_4) &= ((g,g)\cdot f\Pi_\alpha)(x_1,x_2,y)\otimes((g,g)\cdot f\Pi_{\tilde\alpha})(x_3,x_4,y)\cdot \\ &\qquad \langle\phi(y)\otimes\phi(y')\rangle|_{y=x_2,y'=x_4} \\ &= f(x_1,x_2,y)\otimes f(x_3,x_4,y')\cdot \\ &\qquad \langle(g\cdot\phi)(y)\otimes(g\cdot\phi)(y')\rangle|_{y=x_2,y'=x_4} \\ &= W^\alpha(x_1,\dots,x_4),\end{aligned}$$

so both the 4-point functions and the CPWs (going from the noncompact to the induced picture) are contained in the set

$$\mathcal{V}^G := \left(\mathrm{Ind}_Q^G(\pi_1)\otimes\cdots\otimes\mathrm{Ind}_Q^G(\pi_4)\right)^G.$$

This set does not have a group action since we're only considering invariants. However, our algebra action of $U(\mathfrak{g})^{\otimes 4}$ partially survives: let $f: G^4\to V_1\otimes\cdots\otimes V_4$ satisfy

$$f(g_1p_1,\dots,g_4p_4) = \pi_1(p_1^{-1})\otimes\cdots\otimes\pi_4(p_4^{-1})f(g_1,\dots,g_4)$$

($g_1,\dots,g_4\in G, p_1,\dots,p_4\in Q$), then we can act with $U(\mathfrak{g})^{\otimes 4}$ by

$$((\xi_1,\dots,\xi_4)\cdot f)(g_1,\dots,g_4) = \frac{\mathrm{d}}{\mathrm{d}t_1}\cdots\frac{\mathrm{d}}{\mathrm{d}t_4}\, f(\exp(-t_1\xi_1)g_1,\dots,\exp(-t_4\xi_4)g_4)\Big|_{t=0}$$

($\xi_i\in\mathfrak{g}$). In case f is G-invariant, i.e. an element of $\mathcal{V}^G$, we have

$$(g\cdot(q\cdot f))(g_1,\dots,g_4) = (\mathrm{Ad}(g)(q)\cdot f)(g_1,\dots,g_4),$$

so if we have $\mathrm{Ad}(g)(q)=q$, then $q\cdot f\in\mathcal{V}^G$ as well. Thus we have an action of $(U(\mathfrak{g})^{\otimes 4})^G$ on $\mathcal{V}^G$. Let $\Delta: U(\mathfrak{g})\to U(\mathfrak{g})^{\otimes 2}$ be the comultiplication of the Hopf algebra $U(\mathfrak{g})$, given by

$$\Delta(1)=1,\qquad \Delta(\xi)=1\otimes\xi+\xi\otimes 1$$

for $\xi\in\mathfrak{g}$. For any $c\in Z(U(\mathfrak{g}))$, we then have $\Delta(c)\otimes\Delta(1)\in(U(\mathfrak{g})^{\otimes 4})^G$, hence this element can act on our CPWs. If χ_α is the infinitesimal character of the irrep $\mathcal{V}_\alpha = \mathrm{Ind}_G^Q(\pi_\alpha)$, then we obtain

$$\begin{aligned}(\Delta(c)\otimes\Delta(1))\cdot W^\alpha &= (\Delta(c)\otimes\Delta(1))\cdot(f(\cdot,\cdot,y)\Pi_\alpha)\otimes(f(\cdot,\cdot,y')\Pi_{\tilde\alpha}\langle\psi(y)\otimes\psi(y')\rangle) \\ &= (f(\cdot,\cdot,y)\Pi_\alpha)\otimes(f(\cdot,\cdot,y')\Pi_{\tilde\alpha}\langle(c\cdot\psi)(y)\otimes(1\cdot\psi)(y')\rangle \\ &= (f(\cdot,\cdot,y)\Pi_\alpha)\otimes(f(\cdot,\cdot,y')\Pi_{\tilde\alpha}\langle\chi(c)\psi(y)\otimes\psi(y')\rangle \\ &= \chi(c)W^\alpha. \end{aligned}\tag{1.8}$$

This equation is called the *Casimir equation* and the search for CPWs can be reinterpreted as diagonalising the action of the Casimir elements.

2. Useful Tools

2.1. Action on Invariants

Already in the introduction we encountered the situation of wanting to describe actions of invariant (differential) operators on invariant functions. This was a bit awkward, so let's explore a better way to do this.

Definition 2.1.1. *Let G be a Lie group, and let $\mathcal{A}$ be an associative algebra on which G acts smoothly (by means of algebra homomorphisms). An $(\mathcal{A}, G)$-module V is a smooth G-module that is also an $\mathcal{A}$-module such that*

$$g \cdot (x \cdot (g^{-1} \cdot v)) = (g \cdot x) \cdot v$$

for all $v \in V, x \in \mathcal{A}, g \in G$, i.e. such that the G-action on $\mathcal{A}$ corresponds to the adjoint action of $GL(V)$ and $\mathrm{End}(V)$.

Example 2.1.2. *Any smooth G-module is a $(U(\mathfrak{g}), G)$-module, where $g \cdot x := \mathrm{Ad}(x)$ for $x \in \mathfrak{g}$, which is then continued as an algebra automorphism.*

Now for the workhorse of this section:

Lemma 2.1.3. *Let V be an $(\mathcal{A}, G)$-module, then V^G is an $\mathcal{A}^G$-module.*

Proof. Let $v \in V^G, x \in \mathcal{A}^G, g \in G$. Then a priori, $x \cdot v \in V$, but in fact

$$g \cdot (x \cdot v) = g \cdot (x \cdot (g^{-1} g \cdot v)) = (g \cdot x) \cdot (g \cdot v) = x \cdot v,$$

so $x \cdot v \in V^G$. $\square$

Lemma 2.1.4. *Let V be an $(\mathcal{A}, G)$-module, let $\phi : H \to G$ be a Lie algebra homomorphism. Then V is also a $(\mathcal{A}, H)$-module, where the actions of H on V and $\mathcal{A}$ are induced by ϕ.*

Example 2.1.5. *Let $(\pi_1, V_1), \ldots, (\pi_n, V_n)$ be finite-dimensional smooth representations of a subgroup $Q \leq G$ of a Lie group G. Let $L := \mathfrak{g} \otimes \mathbb{C}$ be the complexified Lie algebra of G. Then $\mathcal{V} := \mathrm{Ind}_{Q^n}^{G^n}(\pi_1 \otimes \cdots \otimes \pi_n)$ (smooth or distributional, see Section 4) is a G^n-module*

$$((g_1, \ldots, g_n) \cdot f)(x_1, \ldots, x_n) = f(g_1^{-1} x_1, \ldots, g_n^{-1} x_n)$$

whose G^n-action can be differentiated. As a consequence, it is also a $(U(L)^{\otimes n}, G^n)$-module. Via the map $\Delta_n : G \to G^n, g \mapsto (g, \cdots, g)$ we can make $\mathcal{V}$ into a $(U(L)^{\otimes n}, G)$-module as well (using Lemma 2.1.4).

According to Lemma 2.1.3, the set $\mathcal{V}^G$ of invariants is then a $(U(L)^{\otimes n})^G$-module.

This is what we used in Section 1.6.

2.2. Clifford Algebras

Since we will be dealing with the Dirac operator later, let us now recap some facts (and fix conventions for) Clifford algebras. Throughout this section let V be a $\mathbb{C}$-vector space with symmetric bilinear form η.

Definition 2.2.1. *Define*

$$\mathrm{C}\ell(V,\eta) := \mathrm{C}\ell(V) := T(V)/\langle x \otimes x + \eta(x,x) | x \in V\rangle$$

($T(V)$ the tensor algebra), i.e the associative unital algebra generated by V, subject to the relations $x^2 = -\eta(x,x)$ ($x \in V$), or equivalently

$$\{x,y\} = -2\eta(x,y) \qquad (x,y \in V).$$

This algebra is called the Clifford algebra*.*

Definition 2.2.2. *(a) A vector $v \in V$ is* isotropic *or* null *if $\eta(v,v) = 0$.*

(b) A subspace $U \subseteq V$ is called isotropic *if all of its vectors are isotropic.*

(c) Two subspaces U, U' are called dual *if the restriction $\eta|_{U\times U'}$ is a dual pairing.*

Lemma 2.2.3. *Let η be non-degenerate and V finite-dimensional. Then there are $U, \tilde{U}, Z \leq V$ subspaces such that*

(a) $V = U \oplus \tilde{U} \oplus Z$

(b) $U, \tilde{U}$ are isotropic

(c) $U, \tilde{U}$ are dual

(d) $\dim(Z) \leq 1$

(e) $Z \perp U, \tilde{U}$.

Such a decomposition is called an isotropic decomposition*.*

Proof. In case $\dim(V) = 2n+1$ is odd, pick $v \in V$ with $\eta(v,v) = 1$. Then apply this lemma to the even case of $(\mathbb{C}v)^\perp$. It can be decomposed as $U \oplus \tilde{U}$ (since $U, \tilde{U}$ are dual, they have to have the same dimension. Consequently, Z cannot be 1-dimensional here). Thus, we have

$$V = U \oplus \tilde{U} \oplus \mathbb{C}v,$$

which satisfies the desired properties.

It is therefore without loss of generality that we can assume $\dim(V) = 2n$ is even. Let $v_1, \ldots, v_{2n} \in V$ be an orthonormal basis (it exists since η is nondegenerate, and $\mathbb{C}$ is algebraically closed and not of characteristic 2), then define

$$u_i := \frac{v_{2i-1} + iv_{2i}}{\sqrt{2}}, \qquad \tilde{u}_i := \frac{v_{2i-1} - iv_{2i}}{\sqrt{2}}.$$

We have

$$\eta(u_i, u_j) = \frac{1}{2}(\eta(v_{2i-1}, v_{2j-1}) - \eta(v_{2i}, v_{2j})) = 0$$
$$\eta(\tilde{u}_i, \tilde{u}_j) = \frac{1}{2}(\eta(v_{2i-1}, v_{2j-1}) - \eta(v_{2i}, v_{2j})) = 0$$
$$\eta(u_i, \tilde{u}_j) = \frac{1}{2}(\eta(v_{2i-1}, v_{2j-1}) + \eta(v_{2i}, v_{2j})) = \delta_{ij}.$$

The sets $U := \operatorname{span}\{u_1, \ldots, u_n\}$ and $\tilde{U} := \operatorname{span}\{\tilde{u}_1, \ldots, \tilde{u}_n\}$ are then isotropic and dual. Furthermore,

$$V = U \oplus \tilde{U}$$ □

Definition 2.2.4. *For $u \in V$ define the following endomorphisms of $\bigwedge V$:*

$$\epsilon_u(v_1 \wedge \cdots \wedge v_r) := u \wedge v_1 \wedge \cdots \wedge v_r$$
$$\iota_u(v_1 \wedge \cdots \wedge v_r) := \sum_{i=1}^{r} (-1)^i \eta(u, v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_r,$$

the outer *and* inner product *with u, respectively.*

Proposition 2.2.5. *For $u, v \in V$ we have*

$$\epsilon_u^2 = 0$$
$$\iota_u^2 = 0$$
$$\{\epsilon_u, \iota_v\} = -\eta(u, v) \operatorname{id}.$$

In particular, ϵ's anticommute and ι's anticommute.

Proof. (a) Let $x \in \bigwedge V$, then

$$\epsilon_u \epsilon_u x = u \wedge x = u \wedge u \wedge x = 0.$$

(b) Define

$$a_i(j) := \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

so that a_i is the unique strictly monotonic map $\{1, \ldots, r-1\} \to \{1, \ldots, r\}$ that skips i (face map). Then,

$$a_i \circ a_j = \begin{cases} a_j a_{i-1} & j < i \\ a_{j+1} a_i & j \geq i. \end{cases}$$

Using these face maps we can rewrite ι_u as

$$\iota_u(v_1 \wedge \cdots \wedge v_r) = \sum_{i=1}^{r} (-1)^i B(u, v_i) v_{a_i(1)} \wedge \cdots \wedge v_{a_i(r-1)}.$$

We can now expand ι_u^2 as a double sum over i, j, split up the cases $j < i$ and $j \geq i$, exchange face maps in the second sum, reindex and switch summation order (in the second sum):

$$
\begin{aligned}
\iota_u \iota_u (v_1 \wedge \cdots \wedge v_r) &= \sum_{i=1}^{r} \sum_{j=1}^{r-1} (-1)^{i+j} B(u, v_i) B(u, v_{a_i(j)}) v_{a_i(a_j(1))} \wedge \cdots \wedge v_{a_i(a_j(r-2))} \\
&= \sum_{i=1}^{r} \sum_{j=1}^{i-1} (-1)^{i+j} B(u, v_i) B(u, v_j) v_{a_i(a_j(1))} \wedge \cdots \wedge v_{a_i(a_j(r-2))} \\
&\quad + \sum_{i=1}^{r} \sum_{j=i}^{r-1} (-1)^{i+j} B(u, v_i) B(u, v_{j+1}) v_{a_{j+1}(a_i(1))} \wedge \cdots \wedge v_{a_{j+1}(a_i(r-2))} \\
&= \sum_{i=1}^{r} \sum_{j=1}^{i-1} (-1)^{i+j} B(u, v_i) B(u, v_j) v_{a_i(a_j(1))} \wedge \cdots \wedge v_{a_i(a_j(r-2))} \\
&\quad + \sum_{i=1}^{r} \sum_{j=i+1}^{r} (-1)^{i+j-1} B(u, v_i) B(u, v_j) v_{a_j(a_i(1))} \wedge \cdots \wedge v_{a_j(a_i(r-2))} \\
&= \sum_{i=1}^{r} \sum_{j=1}^{i-1} (-1)^{i+j} B(u, v_i) B(u, v_j) v_{a_i(a_j(1))} \wedge \cdots \wedge v_{a_i(a_j(r-2))} \\
&\quad + \sum_{i=1}^{r} \sum_{j=1}^{i-1} (-1)^{i+j-1} B(u, v_i) B(u, v_j) v_{a_i(a_j(1))} \wedge \cdots \wedge v_{a_i(a_j(r-2))} \\
&= 0.
\end{aligned}
$$

(c) Note that

$$\iota_v(v_1 \wedge x) = -B(v, v_1)x - v_1 \wedge \iota_v(x)$$

for $x \in \bigwedge V$, so that

$$
\begin{aligned}
\iota_v \epsilon_u x &= \iota_v(u \wedge x) \\
&= -B(v, u)x - u \wedge \iota_v(x) \\
&= -B(v, u)x - \epsilon_u \iota_v x,
\end{aligned}
$$

hence $\{\epsilon_u, \iota_v\}x = -\eta(u, v)x$.

□

Corollary 2.2.6. *Let $V = U \oplus \tilde{U} \oplus Z$ be an isotropic decomposition. Define $S := \bigwedge U$.*

(a) For all $\tilde{u} \in \tilde{U}$, the map $\iota_{\tilde{u}}$ restricts to an endomorphism of S

(b) For all $u \in U$, the map ϵ_u restricts to an endomorphism of S.

Proposition 2.2.7. *Let $V = U \oplus \tilde{U} \oplus Z$ be an isotropic decomposition, define $S = \bigwedge U$ as before, and let*

$$S = S^+ \oplus S^- = \bigoplus_{i=0}^{\infty} \bigwedge^{2i} U \oplus \bigoplus_{i=0}^{\infty} \bigwedge^{2i+1} U$$

be the decomposition into odd and even parts. Define

$$m_\pm \in \mathrm{End}(S), \qquad S^+ \oplus S^- \ni s^+ + s^- \mapsto \pm i(s^+ - s^-),$$

then both m_+ and m_- anticommute with $\epsilon_u, \iota_{\tilde u}$ ($u \in U, \tilde u \in \tilde U$), and satisfy $m_\pm^2 = -1$.

Proof. Note that both ϵ_u and $\iota_{\tilde u}$ change the degree by one, hence they change the parity. For $x \in S^\epsilon$ ($\epsilon \in \{\pm 1\}$) we therefore have $m_\pm x = \pm i\epsilon x$ and

$$m_\pm \iota_{\tilde u} x = \mp i\epsilon\iota_{\tilde u} x, \qquad m_\pm \epsilon_u x = \mp i\epsilon\epsilon_u x,$$

whence

$$\begin{aligned}(m_\pm \iota_{\tilde u} + \iota_{\tilde u} m_\pm)x &= \mp i\epsilon\iota_{\tilde u} x \pm i\epsilon\iota_{\tilde u} x = 0\\ (m_\pm \epsilon_u + \epsilon_u m_\pm)x &= \mp i\epsilon\epsilon_u x \pm i\epsilon\epsilon_u x = 0.\end{aligned}$$

Furthermore, since the two eigenvalues of $m_\pm$ are $\pm i$, both endomorphisms square to -1. □

Definition 2.2.8. *Assume $V = U \oplus \tilde U$ is an isotropic decomposition. The vector space $S = \bigwedge U$ is called the* spin module *for $\mathrm{C}\ell(V)$, and $\mathrm{C}\ell(V)$ acts as follows:*

$$U \oplus \tilde U \ni (u + \tilde u) \cdot x := (\epsilon_u + 2\iota_{\tilde u})x.$$

Assume $V = U \oplus \tilde U \oplus \mathbb{C}z$ is an isotropic decomposition with $\eta(z, z) = 1$. The vector space $S = \bigwedge U$ can be turned into two different $\mathrm{C}\ell(V)$-modules S_1 and S_2, also called spin modules*, by*

$$U \oplus \tilde U \oplus \mathbb{C}z \ni (u + \tilde u + az) \cdot x := (\epsilon_u + 2\iota_{\tilde u} + am_\pm)x.$$

Proposition 2.2.9. *Both S in the even case, and S_1, S_2 in the odd case are $\mathrm{C}\ell(V)$-modules.*

Proof. "even": It suffices to show that the endomorphism on the right-hand side has the correct square. Let $u \in U, \tilde u \in \tilde U$, then

$$\begin{aligned}(\epsilon_u + 2\iota_{\tilde u})^2 &= \epsilon_u^2 + 4\iota_{\tilde u}^2 + 2\{\epsilon_u, \iota_{\tilde u}\}\\ &= -2\eta(u, \tilde u)\,\mathrm{id}\\ &= -\eta(u + \tilde u, u + \tilde u)\,\mathrm{id}\end{aligned}$$

by Proposition 2.2.5.
"odd": Let $u \in U, \tilde u \in \tilde U, a \in \mathbb{C}$, then

$$\begin{aligned}(\epsilon_u + 2\iota_{\tilde u} + am_\pm)^2 &= (\epsilon_u + 2\iota_{\tilde u})^2 + a^2 m_\pm^2 + a\{\epsilon_u, m_\pm\} + 2a\{\iota_{\tilde u}, m_\pm\}\\ &= -\eta(u + \tilde u, u + \tilde u)\,\mathrm{id} - a^2\,\mathrm{id}\\ &= -\eta(u + \tilde u + az, u + \tilde u + az)\,\mathrm{id}\end{aligned}$$

by the previous case and by Proposition 2.2.7. □

Theorem 2.2.10. *Let η be non-degenerate and V finite-dimensional. Then*

$$\mathrm{C}\ell(V) \cong \begin{cases} \mathrm{End}(S) & 2|\dim(V) \\ \mathrm{End}(S_1) \oplus \mathrm{End}(S_2) & 2|(\dim(V)-1) \end{cases}.$$

Proof. For the even case, see [HP06, lemma 2.2.4], and for the odd case [HP06, section 2.2.7]. □

Proposition 2.2.11. *Let $v_1, \ldots, v_n$ be an orthonormal basis of V and write $F_{ij} \in \mathfrak{so}(V)$ for the map mapping $v_j \mapsto v_i$, $v_i \mapsto -v_j$, and leaving all other basis vectors invariant. It is well-known that $(F_{ij})_{1\le i<j\le n}$ is a basis of $\mathfrak{so}(V)$. Let*

$$j : \mathfrak{so}(V) \to \mathrm{C}\ell(V), \qquad F_{ij} \mapsto \frac{1}{2} v_j v_i,$$

then j is a Lie algebra homomorphism and for $f \in \mathfrak{so}(V), v \in V$ we have

$$f(v) = [j(f), v].$$

Proof. We have

$$[F_{ij}, F_{kl}] = \delta_{jk} F_{il} + \delta_{il} F_{jk} - \delta_{ik} F_{jl} - \delta_{jl} F_{ik}$$

on one hand, and

$$\begin{aligned} \frac{1}{4}[v_j v_i, v_l v_k] &= \frac{1}{4}(v_j v_i v_l v_k - v_l v_k v_j v_i) \\ &= \frac{1}{4}(-2\delta_{il} v_j v_k - v_j v_l v_i v_k + 2\delta_{jk} v_l v_i + v_l v_j v_k v_i) \\ &= \frac{1}{4}(2\delta_{il} v_k v_j + 4\delta_{il}\delta_{jk} + 2\delta_{lj} v_i v_k + v_l v_j v_i v_k + 2\delta_{jk} v_l v_i - 2\delta_{ik} v_l v_j - v_l v_j v_i v_k) \\ &= \frac{1}{2}(\delta_{il} v_k v_j + 2\delta_{il}\delta_{jk} + \delta_{lj} v_i v_k + \delta_{jk} v_l v_i - \delta_{ik} v_l v_j) \\ &= \delta_{il} j(F_{jk}) + \delta_{jk} j(F_{il}) - \delta_{jl} j(F_{ik}) - \delta_{ik} j(F_{jl}) \\ &= j([F_{ij}, F_{kl}]). \end{aligned}$$

Note that

$$\delta_{il} v_k v_j + \delta_{jk} v_l v_i + 2\delta_{il}\delta_{jk} = \delta_{il} j(F_{jk}) + \delta_{jk} j(F_{il})$$

because in case $i = l$ and $j = k$ both sides are zero, and otherwise the double δ term vanishes.

Furthermore, we have

$$\begin{aligned} j(F_{ij}) v_k - v_k j(F_{ij}) &= \frac{1}{2}(v_j v_i v_k - v_k v_j v_i) \\ &= \frac{1}{2}(-2\delta_{ik} v_j - v_j v_k v_i + 2\delta_{jk} v_i + v_j v_k v_i) \\ &= \delta_{jk} v_i - \delta_{ik} v_j \\ &= F_{ij}(v_k). \end{aligned}$$

□

Definition 2.2.12. *The Lie algebra homomorphism* $j : \mathfrak{so}(V) \to \mathrm{C}\ell(V)$ *is called the* Chevalley embedding*.*

Proposition 2.2.13. *Let* $v_1, \dots, v_n \in V$ *be an orthonormal basis, let* $f \in \mathfrak{so}(V)$*, then*

$$j(f) = \frac{1}{4} \sum_{i,j=1}^{n} \eta(f(v_i), v_j) v_i v_j.$$

Proof. Let $f = \frac{1}{2} \sum_{i,j=1}^{n} a^{ij} F_{ij}$ for a^{ij} antisymmetric in i, j, then

$$\begin{aligned}
\frac{1}{4} \sum_{i,j=1}^{n} \eta(f(v_i), v_j) v_i v_j &= \frac{1}{8} \sum_{i,j,k,l=1}^{n} a^{kl} \eta(F_{kl}(v_i), v_j) v_i v_j \\
&= \frac{1}{8} \sum_{i,j,k,l=1}^{n} a^{kl} \eta(\delta_{il} v_k - \delta_{ik} v_l, v_j) v_i v_j \\
&= \frac{1}{8} \sum_{i,j,k,l=1}^{n} a^{kl} (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) v_i v_j \\
&= \frac{1}{4} \sum_{i,j=1}^{n} a^{ij} v_j v_i \\
&= \frac{1}{2} \sum_{i,j=1}^{n} a^{ij} j(F_{ij}) \\
&= j(f).
\end{aligned}$$

□

2.3. Central Characters

This section is mostly inspired by [Bou58, chap. VIII, §3, no. 2]. We will discuss a very general (algebraic) version of Schur's lemma, and how it implies the existence of a map called the *(central) character* that helps classify irreducible representations (/simple modules).

Throughout this section let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra.

Definition 2.3.1. *A left* $\mathcal{A}$*-module* M *is called simple if* $M \neq 0$ *and its only* $\mathcal{A}$*-submodules are* 0 *and* M *itself.*

Proposition 2.3.2. *Let* $\phi \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ *be a linear map between* $\mathcal{A}$*-modules.*

(a) If M *is simple then* ϕ *is either 0 or injective.*

(b) If N *is simple then* ϕ *is either 0 or surjective.*

Proof. (a) Let $v \in \ker(\phi)$ and $a \in \mathcal{A}$, then $\phi(av) = a\phi(v) = a0 = 0$, so $av \in \ker(\phi)$. This (together with the fact that ϕ is a $\mathbb{C}$-linear map, so $\ker(\phi) \subseteq M$ is a sub-vector space) shows that $\ker(\phi) \leq M$ is an $\mathcal{A}$-submodule. Since M is simple, we either have $\ker(\phi) = M$, in which case $\phi = 0$, or $\ker(\phi) = 0$, in which case ϕ is injective.

(b) Let $v \in M$ and $a \in \mathcal{A}$, then $a\phi(v) = \phi(av) \in \mathrm{im}(\phi)$, so $\mathrm{im}(\phi) \leq N$ is an $\mathcal{A}$-submodule. Due to N's simplicity, either $\mathrm{im}(\phi) = 0$, in which case $\phi = 0$, or $\mathrm{im}(\phi) = N$, in which case ϕ is surjective.

□

Corollary 2.3.3. *Let M be a simple $\mathcal{A}$-module. Then $\mathrm{End}_{\mathcal{A}}(M)$ is a skew field.*

Proof. $\mathrm{End}_{\mathcal{A}}(M)$ is an associative $\mathbb{C}$-algebra, so all we need to do is to show that every nonzero element of $\mathrm{End}_{\mathcal{A}}(M)$ is invertible. This follows from Proposition 2.3.2. □

Lemma 2.3.4. *Let E be a (not necessarily finite-dimensional) skew field containing $\mathbb{C}$ in its centre. If $E \neq \mathbb{C}$, we have $[E : \mathbb{C}] \geq \beth_1$ (the cardinality of $\mathbb{C}$).*

Proof. In case $E \neq \mathbb{C}$, there is an element $x \in E \setminus \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, x is transcendental, hence the family of elements $((x - \alpha)^{-1})_{\alpha \in \mathbb{C}} \in E$ is $\mathbb{C}$-linearly independent, hence the dimension of E as $\mathbb{C}$-vector space is at least $\beth_1$. □

Theorem 2.3.5. *Let M be a simple $\mathcal{A}$-module such that $\dim_{\mathbb{C}}(M) < \beth_1$. Then $\mathrm{End}_{\mathcal{A}}(M) = \mathbb{C}\,\mathrm{id}$.*

Proof. By Corollary 2.3.3, $\mathrm{End}_{\mathcal{A}}(M)$ is a skew field, which contains $\mathbb{C}$ in its centre. Thus, by Lemma 2.3.4, we either have $\mathrm{End}_{\mathcal{A}}(M) = \mathbb{C}$ or $[\mathrm{End}_{\mathcal{A}}(M) : \mathbb{C}] \geq \beth_1$. If the second is true, we can conclude

$$\dim_{\mathbb{C}}(M) = [\mathrm{End}_{\mathcal{A}}(M) : \mathbb{C}] \dim_{\mathrm{End}_{\mathcal{A}}(M)}(M) \geq \beth_1,$$

which contradicts our assumption. Therefore, $\mathrm{End}_{\mathcal{A}}(M) = \mathbb{C}$. □

Corollary 2.3.6 (Schur, Dixmier)**.** *Let M, N be simple $\mathcal{A}$-modules.*

(a) We have $\mathrm{Hom}_{\mathcal{A}}(M, N) = 0$ or M, N are isomorphic.

(b) If one of M, N is known to have $\mathbb{C}$-dimension less than $\beth_1$, this specialises to

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{A}}(M, N)) = \begin{cases} 0 & M \ncong N \\ 1 & M \cong N. \end{cases}$$

Proof. (a) Let $\phi \in \mathrm{Hom}_{\mathcal{A}}(M, N)$. If $\phi \neq 0$, Proposition 2.3.2 implies that ϕ is bijective, hence an isomorphism.

(b) It remains to show that if $M \cong N$, the dimension of $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is at most one. Let $\phi, \psi \in \mathrm{Hom}_{\mathcal{A}}(M, N)$. If one of them is zero, they are evidently linearly dependent, so assume without loss of generality that they are nonzero. By Proposition 2.3.2, they are both isomorphisms. Then $\phi^{-1} \circ \psi \in \mathrm{End}_{\mathcal{A}}(M)$ is a multiple of the identity by Theorem 2.3.5, say λ. Then $\psi - \lambda\phi = 0$, hence ψ, ϕ are linearly dependent.

□

Theorem 2.3.7. *Let M be a simple $\mathcal{A}$-module whose $\mathbb{C}$-dimension is less than $\beth_1$. There exists an algebra homomorphism $\chi : Z(\mathcal{A}) \to \mathbb{C}$ such that for all $z \in Z(\mathcal{A}), v \in M$ we have*

$$zv = \chi(z)v.$$

Proof. Define $\chi : Z(\mathcal{A}) \to \mathrm{End}_{\mathcal{A}}(M)$ to be the left action of $\mathcal{A}$ (restricted to its centre). This is well-defined because for $z \in Z(\mathcal{A}), a \in \mathcal{A}, v \in M$ we have

$$\chi(z)(av) = zav = azv = a\chi(z)(v),$$

thus $\chi(z)$ is an $\mathcal{A}$-endomorphism. Because of the associativity in the definition of a module, we have $\chi(z)\chi(w) = \chi(zw)$, making χ into an algebra homomorphism.

By Theorem 2.3.5, χ maps to the scalar multiples of the identity, so we can compose with the $\mathbb{C}$-algebra morphism mapping $\mathrm{id} \mapsto 1$. □

The map χ is called the *central character* of M. Write $\hat{\mathcal{A}}$ for the set of algebra homomorphisms $Z(\mathcal{A}) \to \mathbb{C}$.

Corollary 2.3.8. *Let $\mathcal{A}$ have at most countable $\mathbb{C}$-dimension. Then every simple $\mathcal{A}$-module has a central character.*

Proof. Let M be a simple $\mathcal{A}$-module. Let $0 \neq v \in M$, then v is a cyclic vector (otherwise, $\mathcal{A}v$ would be a nontrivial submodule, contradicting M's simplicity). Thus,

$$\dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}(\mathcal{A}v) \leq \dim_{\mathbb{C}}(\mathcal{A}) \leq \beth_0 < \beth_1.$$

We can thus apply Theorem 2.3.7. □

Corollary 2.3.9. *Let L be an at most countable-dimensional complex Lie algebra. Any simple $U(L)$-module has a central character.*

Proof. $U(L)$ has countable dimension over $\mathbb{C}$ (unless $L = 0$, then it even has finite dimension), so we can apply Corollary 2.3.8. □

Definition 2.3.10. *Let $\chi \in \hat{\mathcal{A}}$, let M be an $\mathcal{A}$-module, then define*

$$M[\chi] := \{v \in M \mid \forall z \in Z(\mathcal{A}) : zv = \chi(z)v\},$$

the χ-isotypic component *of M.*

Proposition 2.3.11. *Let $\chi \in \hat{\mathcal{A}}$, then the mapping*

$$M \mapsto M[\chi], \qquad f \in \mathrm{Hom}_{\mathcal{A}}(M, N) \mapsto f|_{M[\chi]}$$

defines an endofunctor in the category of $\mathcal{A}$-modules that commutes with direct sums.

Proof. We need to show that any $\mathcal{A}$-linear maps maps isotypic components to isotypic components. Let $\phi : M \to N$ be $\mathcal{A}$-linear, let $z \in Z(\mathcal{A}), v \in M[\chi]$, then

$$z\phi(v) = \phi(zv) = \phi(\chi(z)v) = \chi(z)\phi(v),$$

hence $\phi(v) \in N[\chi]$. Evidently composition survives restriction.

Let $(M_i)_{i\in I}$ be a family of modules, we show that

$$\left(\bigoplus_{i\in I} M_i\right)[\chi] = \bigoplus_{i\in I} M_i[\chi].$$

"$\supseteq$": Let $x_i \in M_i[\chi]$, only finitely many x_i nonzero. Then for $z \in Z(\mathcal{A})$ we have

$$z\sum_{i\in I} x_i = \sum_{i\in I} zx_i = \sum_{i\in I} \chi(z)x_i = \chi(z)\sum_{i\in I} x_i,$$

so $\sum_{i\in I} x_i$ lies in the χ-isotypic component of $\bigoplus_{i\in I} M_i$.
"$\subseteq$": Let $x_i \in M_i$, only finitely many x_i nonzero, such that

$$\sum_{i\in I} zx_i = z\sum_{i\in I} x_i = \chi(z)\sum_{i\in I}\sum_{i\in I} \chi(z)x_i$$

for all $z \in Z(\mathcal{A})$. Subtracting the left and the right side from each other, we get

$$0 = \sum_{i\in I}(z - \chi(z))x_i.$$

Since the $i \in I$-term lies in M_i, and the sum is direct, we have $(z - \chi(z))x_i = 0$ $(i \in I)$, whence $x_i \in M_i[\chi]$. □

Proposition 2.3.12. *Let M be a semisimple $\mathcal{A}$-module whose simple components have central characters, then*

$$M = \bigoplus_{\chi\in\hat{\mathcal{A}}} M[\chi].$$

This is known as the isotypic decomposition.

Proof. Let N be a simple $\mathcal{A}$-module with central character, then $N[\chi] = N$ or 0, depending on whether χ is N's central character or not. Thus,

$$M = \bigoplus_{i\in I} M_i$$

where the M_i are simple modules. For $\chi \in \hat{\mathcal{A}}$ write I_χ for the set of indices such that χ is M_i's central character. Since every M_i has a central character, the I_χ partition I, so that

$$\begin{aligned}\bigoplus_{\chi\in\hat{\mathcal{A}}} M[\chi] &= \bigoplus_{\chi\in\hat{\mathcal{A}}}\bigoplus_{i\in I} M_i[\chi] \\ &= \bigoplus_{\chi\in\hat{\mathcal{A}}}\bigoplus_{i\in I_\chi} M_i \\ &= \bigoplus_{i\in I} M_i = M.\end{aligned}$$

□

3. Structure of G

With the setup and the tools out of the way, we can now get started with the actual hamonic analysis perspective on conformal blocks. For that let's have a closer look at the group we will be dealing with for the rest of this thesis.

Let $d = p + q > 2$ with $p \geq q$, and fix $G := SO(p + 1, q + 1)_0$. For our purposes, we will mainly be interested in the cases $q = 0$ (Euclidean CFT) and $q = 1$ (Lorentzian CFT), but a lot of the structure theory can be done in more generality[1]. Write η for the standard symmetric bilinear form on $\mathbb{R}^{d+2}$ of signature $(p+1, q+1)$ and interchangeably for the matrix $\operatorname{diag}(1, \dots, 1, -1, \dots, -1)$ that generates this bilinear form. Note that we will also (continue to) use the Einstein summation convention where Greek indices run from 0 to $d + 1$ or from 1 to d. In particular, η will be used to raise and lower indices.

By [Kna96, section VII.2, example 2], (G, K, θ, B) is a reductive group, where

$$\begin{aligned} K &= G \cap SO(d + 2) = SO(p + 1) \times SO(q + 1) \\ \theta(x) &= -x^T = \eta x \eta \\ B(x, y) &= \operatorname{tr}(xy). \end{aligned}$$

3.1. Restricted Root Spaces

We now work out the restricted root space decomposition of $\mathfrak{g}$, en route to finding its Iwasawa decomposition. We begin by defining a basis. Starting with

$$(E_{\mu\nu})^\rho{}_\sigma := \delta^\mu_\rho \eta_{\nu\sigma}$$

and

$$F_{\mu\nu} := E_{\mu\nu} - E_{\nu\mu}.$$

This is a basis for $\mathfrak{g}$.

Proposition 3.1.1. *We have*

(a) $[F_{\mu\nu}, F_{\rho\sigma}] = \eta_{\nu\rho} F_{\mu\sigma} + \eta_{\mu\sigma} F_{\nu\rho} - \eta_{\nu\sigma} F_{\mu\rho} - \eta_{\mu\rho} F_{\nu\sigma}$

(b) $F_{\mu\nu} = -F_{\nu\mu}$

(c) $F^T_{\mu\nu} = F_{\mu\nu}$ *if* $\mu \leq p < \nu$ *or vice-versa, and* $F^T_{\mu\nu} = -F_{\mu\nu}$ *if* $\mu, \nu \leq p$ *or* $p < \mu, \nu$

[1] Note that the case $q = 1$ somewhat differs from the others, as in that case G's fundamental group is $\mathbb{Z}$, so that its universal cover does not have finite centre, and is hence not of Harish-Chandra type. This makes its representation theory harder to grasp, and is one of the reasons we're not looking at universal covers in this thesis.

Proof. (a) We start by calculating products of E's:

$$\begin{aligned}(E_{\mu\nu}E_{\rho\sigma})^{\alpha}{}_{\beta} &= (E_{\mu\nu})^{\alpha}{}_{\gamma}(E_{\rho\sigma})^{\gamma}{}_{\beta} \\ &= \delta^{\alpha}_{\mu}\eta_{\nu\gamma}\delta^{\gamma}_{\rho}\eta_{\sigma\beta} \\ &= \eta_{\nu\rho}(E_{\mu\sigma})^{\alpha}{}_{\beta},\end{aligned}$$

hence $E_{\mu\nu}E_{\rho\sigma} = \eta_{\nu\rho}E_{\mu\sigma}$. This shows that

$$[F_{\mu\nu}, F_{\rho\sigma}] = \eta_{\nu\rho}F_{\mu\sigma} + \eta_{\mu\sigma}F_{\nu\rho} - \eta_{\nu\sigma}F_{\mu\rho} - \eta_{\mu\rho}F_{\nu\sigma}.$$

(b) From definition.

(c) If $\eta_{\mu\mu} = \eta_{\nu\nu}$, we have $\delta^{\rho}_{\mu}\eta_{\nu\sigma} = \eta_{\rho\mu}\delta_{\nu\sigma}$, and hence

$$\begin{aligned}(F_{\mu\nu})^{\rho}{}_{\sigma} &= \delta^{\rho}_{\mu}\eta_{\nu\sigma} - \delta^{\rho}_{\nu}\eta_{\mu\sigma} \\ &= \eta_{\rho\mu}\delta^{\sigma}_{\nu} - \eta_{\rho\nu}\delta^{\sigma}_{\mu} \\ &= (F_{\nu\mu})^{\sigma}{}_{\rho} \\ &= (-F^{T}_{\mu\nu})^{\rho}{}_{\sigma}.\end{aligned}$$

This is the case if $\mu\nu \leq p$ or $> p$.

Otherwise, we have $\eta_{\mu\nu} = -\eta_{\nu\nu}$, then we acquire another minus, hence $(F_{\mu\nu})^T = F_{\mu\nu}$.

□

We see that

$$\begin{aligned}\mathfrak{k} &= \operatorname{span}\{F_{\mu\nu} \mid \mu, \nu \leq p \text{ or } p < \mu, \nu\} \\ \mathfrak{p} &= \operatorname{span}\{F_{\mu\nu} \mid \mu \leq p < \nu\}.\end{aligned}$$

For $\mu < \nu$, the matrix $F^{\mu\nu}$ has a 1 at (μ, ν), and ± 1 (depending on if it is contained in $\mathfrak{p}$ or $\mathfrak{k}$) in (ν, μ).

We now pick a maximal commutative subspace of $\mathfrak{p}$. Define $D_i := F^{i,d+1-i}$ ($i = 0, \dots, q$).

Proposition 3.1.2. *The vector space*

$$\mathfrak{a}_{\mathfrak{p}} := \operatorname{span}\{D_0, \dots, D_q\} \leq \mathfrak{p}$$

is a maximal commutative subalgebra.

Proof. From Proposition 3.1.1(a) we know that any D_i commutes with all $F^{\mu\nu}$ with $\{\mu, \nu\} \cap \{i, d+1-i\} = \emptyset$. Consequently, we have $[D_i, D_j] = 0$, i.e. $\mathfrak{a}_{\mathfrak{p}}$ is a commutative subalgebra.

Let now

$$x = \frac{1}{2}a^{\mu\nu}F_{\mu\nu} \in \mathfrak{p}$$

($a^{\mu\nu}$ antisymmetric) commute with all D_i, then

$$\begin{aligned}[D_i, x] &= \frac{1}{2} a^{\mu\nu} (\delta_\mu^{d+1-i} F^i{}_\nu + \delta_\nu^i F^{d+1-i}{}_\mu - \delta_\nu^{d+1-i} F^i{}_\mu - \delta_\mu^i F^{d+1-1}{}_\nu) \\ &= a^{d+1-i}{}_\nu F^{i,\nu} - a^i{}_\nu F^{d+1-i,\nu} \\ &= 0\end{aligned}$$

That $x \in \mathfrak{p}$, implies that $a^{\mu\nu} = 0$ for $\mu, \nu \leq p$ and $\mu, \nu > p$. Thus the first contraction $a^i{}_\nu F^{d+1-1,\nu}$ lies in the $0 \oplus \mathfrak{so}(q+1)$-component of $\mathfrak{k}$, and the second contraction $a_\mu{}^{d+1-i} F^{i,\mu}$ lies in the $\mathfrak{so}(p+1) \oplus 0$-component. Thus, they are linearly independent, and we get $a^\mu{}_{d+1-i} = 0$ for all $\mu \leq p$ and $i \leq q$, hence $a = 0$. Thus, $\mathfrak{a}_\mathfrak{p}$ is indeed maximal. □

Corollary 3.1.3. *$\mathfrak{g}$ has real rank $q+1$.*

Define $\epsilon_i \in \mathfrak{a}_\mathfrak{p}^*$ by $\epsilon_i(D_j) = \delta_{i,j}$.

Proposition 3.1.4. *Then*

$$\mathfrak{g} = \mathfrak{m}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where

$$R := \{\epsilon_i, \epsilon_i \pm \epsilon_j, -\epsilon_i \pm \epsilon_j \mid 0 \leq i < j \leq q\}$$

is of type B_{q+1}.

Proof. From the theory of semisimple Lie groups, e.g. [Kna96, section VI.4] it is well-known that our desired decomposition exists (for R the set of roots) and is direct. So it suffices to find all the roots.

Let $0 \leq i, j \leq q$ and $q < \mu \leq p$, then

$$\begin{aligned}\left[D_i, F^{j\mu} \pm F^{d+1-j,\mu}\right] &= -\eta^{i,j} F^{d+1-i,\mu} \pm \eta^{d+1-i,d+1-j} F^{i,\mu} \\ &= \mp\delta_{i,j} \left(F^{i,\mu} \pm F^{d+1-i,\mu}\right) \\ &= (\mp\epsilon_j)(D_i)(F^{j\mu} \pm F^{d+1-j,\mu}),\end{aligned}$$

so $F^{j\mu} \pm F^{d+1-j,\mu} \in \mathfrak{g}_{\mp\epsilon_j}$.

Let $0 \leq i \neq j, k \leq q$, then

$$\begin{aligned}&\left[D_k, F^{ij} \pm F^{i,d+1-j} + F^{d+1-i,j} \pm F^{d+1-i,d+1-j}\right] \\ =&\eta^{jk} F^{d+1-k,i} - \eta^{ik} F^{d+1-k,j} \mp \eta^{ik} F^{d+1-k,d+1-j} \mp \eta^{d+1-k,d+1-j} F^{ki} \\ &+ \eta^{d+1-i,d+1-k} F^{kj} + \eta^{jk} F^{d+1-k,d+1-i} \pm \eta^{d+1-i,d+1-k} F^{k,d+1-j} \mp \eta^{d+1-j,d+1-k} F^{k,d+1-i} \\ =&(\mp\delta_{jk} - \delta_{ik}) \left(F^{ij} \pm F^{i,d+1-j} + F^{d+1-i,j} \pm F^{d+1-i,d+1-j}\right) \\ =&(-\epsilon_i \mp \epsilon_j)(D_k) \left(F^{ij} \pm F^{i,d+1-j} + F^{d+1-i,j} \pm F^{d+1-i,d+1-j}\right),\end{aligned}$$

so that $F^{ij} \pm F^{i,d+1-j} + F^{d+1-i,j} \pm F^{d+1-i,d+1-j} \in \mathfrak{g}_{-\epsilon_i \mp \epsilon_j}$, and θ applied to it lies in $\mathfrak{g}_{\epsilon_i \pm \epsilon_j}$.

Lastly, for $q < \mu\nu \le p$ we have

$$[D_k, F^{\mu\nu}] = 0,$$

so that $F^{\mu\nu} \in \mathfrak{m}_\mathfrak{p}$.

Thus, we've found the following roots: $\pm\epsilon_i$ $(i = 0, \dots, q)$, $\epsilon_i \pm \epsilon_j$, and $-\epsilon_i \pm \epsilon_j$ (both for $i \neq j$). We can now count dimensions to show that we've already accounted for everything: we know

$$\dim(\mathfrak{g}_{\pm\epsilon_i}) \ge p - q, \qquad \dim(\mathfrak{g}_{\epsilon_i \pm \epsilon_j}), = \dim(\mathfrak{g}_{-\epsilon_i \mp \epsilon_j}) \ge 1$$

and

$$\dim(\mathfrak{m}_\mathfrak{p}) \ge \frac{(p-q)(p-q-1)}{2},$$

so that we have

$$\begin{aligned}
\frac{(d+2)(d+1)}{2} &= \dim(\mathfrak{g}) \\
&= \dim\left(\mathfrak{m}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\right) \\
&= \dim(\mathfrak{m}_\mathfrak{p}) + \dim(\mathfrak{a}_\mathfrak{p}) + \sum_{\alpha \in R} \dim(\mathfrak{g}_\alpha) \\
&\ge \frac{(p-q)(p-q-1)}{2} + q + 1 + 2(q+1)(p-q) + 2(q+1)q \\
&= \frac{(d+2)(d+1)}{2},
\end{aligned}$$

so we have found all roots and all root spaces. □

3.2. Conformal Compactification and Parabolic Subgroup

As mentioned in the introduction, the group G is of interest to us because it is the group of (local) conformal transformations of $\mathbb{R}^{p,q}$. This is proven e.g. in [Sch08, theorem 2.9]. The way this works is that we embed $\mathbb{R}^{p,q}$ into a bigger (compact) manifold $\widehat{\mathbb{R}^{p,q}}$, on which G acts globally. This embedding is called the *conformal compactification* of $\widehat{\mathbb{R}^{p,q}}$.

We define

$$\widehat{\mathbb{R}^{p,q}} := \left\{ (x_0 : \dots : x_{d+1}) \in \mathbb{P}^{d+1}(\mathbb{R}) \,\middle|\, x_\mu x^\mu = \eta(x,x) = 0 \right\},$$

the projectivisation of the subset of (non-zero) isotropic vectors in $\mathbb{R}^{p+1,q+1}$, and the embedding is

$$\iota : R^{p,q} \to \widehat{R^{p,q}}, \qquad v \mapsto (1 - \eta(v,v) : 2v : 1 + \eta(v,v))$$

(we use η for the symmetric bilinear form on $\mathbb{R}^{p,q}$ as well).

Proposition 3.2.1. *We have*

$$\widehat{R^{p,q}} = \iota(\mathbb{R}^{p,q}) \sqcup \{x \in \mathbb{R}^{p,q} \mid \eta(x,x) = 0\} \sqcup \widehat{\mathbb{R}^{p-1,q-1}} = X_1 \sqcup X_2 \sqcup X_3$$

and $\iota(\mathbb{R}^{p,q})$ is dense. In particular, $X_2 \cup X_3 = \partial X_1$ and $X_3 = \partial X_2$.

Proof. Let

$$(x_0 : \cdots : x_{d+1}) \in \widehat{\mathbb{R}^{p,q}}.$$

If $x_0 + x_{d+1} \neq 0$, define $\underline{x} := \frac{(x_1,\ldots,x_d)}{x_0+x_{d+1}}$, then

$$\eta(\underline{x},\underline{x}) = \frac{x_{d+1}^2 - x_0^2}{(x_0 + x_{d+1})^2} = \frac{x_{d+1} - x_0}{x_{d+1} + x_0},$$

hence

$$\begin{aligned}\iota(\underline{x}) &= \left(1 - \frac{x_{d+1} - x_0}{x_{d+1} + x_0} : \frac{2(x_1 : \cdots : x_d)}{x_0 + x_{d+1}} : 1 + \frac{x_{d+1} - x_0}{x_{d+1} + x_0}\right) \\ &= (x_{d+1} + x_0 - x_{d+1} + x_0 : 2x_1 : \cdots : 2x_d : x_{d+1} + x_0 + x_{d+1} - x_0) \\ &= (x_0 : \cdots : x_{d+1}).\end{aligned}$$

Otherwise, we have $x_0 = -x_{d+1}$. If both are nonzero, we can fix $x_0 = -x_{d+1} = 1$, then $x_1, \ldots, x_d$ are fixed and form an isotropic vector in $\mathbb{R}^{p,q}$.

If both are zero, we have an element of $\widehat{\mathbb{R}^{p-1,q-1}}$ via the embedding

$$(x_0 : \cdots : x_{d-1}) \mapsto (0 : x_0 : \cdots : x_{d-1} : 0).$$

This shows the decomposition of $\widehat{\mathbb{R}^{p,q}}$. To see that $\iota(\mathbb{R}^{p,q})$ is dense, note that we can reach all of $\widehat{\mathbb{R}^{p,q}}$ as limits of straight lines in $\mathbb{R}^{p,q}$: Let $u, v \in \mathbb{R}^{p,q}$ ($v \neq 0$), consider the line $u + vt$:

$$\iota(u{+}vt) = \Big(1 - \eta(u,u) - 2t\eta(u,v) - t^2\eta(v,v) : 2u + 2vt : 1 + \eta(u,u) + 2t\eta(u,v) + t^2\eta(v,v)\Big).$$

If $\eta(v,v) \neq 0$, we have

$$\iota(u+vt) = \left(-\frac{1}{t^2} + \frac{\eta(u,u)}{t^2} + \frac{2\eta(u,v)}{t} + 1 : -\frac{2u}{t^2} - \frac{2v}{t} : -\frac{1}{t^2} - \frac{\eta(u,u)}{t^2} - \frac{2\eta(u,v)}{t} - 1\right),$$

which converges to $(1 : 0 : -1)$ as $t \to \infty$.

If v is isotropic, but $\eta(u,v) \neq 0$, we have

$$\iota(u+vt) = \left(\frac{\eta(u,u) - 1}{2t\eta(u,v)} + 1 : -\frac{u}{t\eta(u,v)} - \frac{v}{\eta(u,v)} : \frac{-\eta(u,u) - 1}{2t\eta(u,v)} - 1\right),$$

which converges to $\left(1 : -\frac{v}{\eta(u,v)} : -1\right) \in X_2$ as $t \to \infty$, showing that $X_2 \subseteq \overline{X_1}$

If we replace u by ϵu, we obtain

$$\left(1 : -\frac{v}{\epsilon\eta(u,v)} : -1\right) = \left(\epsilon : -\frac{v}{\eta(u,v)} : -\epsilon\right),$$

which converges to $(0 : -\frac{v}{\eta(u,v)} : 0) \in X_3$ as $\epsilon \to 0$. This shows that $\overline{X_2} = X_2 \sqcup X_3$.

If v is isotropic and $\eta(u, v) = 0$ we have

$$\iota(u + vt) = \left(\frac{1 - \eta(u, u)}{2t} : \frac{u}{t} + v : \frac{1 + \eta(u, u)}{2t}\right),$$

which converges to $(0 : v : 0) \in X_3$ as $t \to \infty$. This shows that $X_2 \subseteq \overline{X_1}$ as well, hence that $\widehat{\mathbb{R}^{p,q}} = \overline{\iota(\mathbb{R}^{p,q})}$. □

Two elements of $\widehat{\mathbb{R}^{p,q}}$ are of particular importance: $(1 : 0 : 1) = \iota(0)$ will be called the *origin*, and $(1 : 0 : -1) = \infty$ will be called the *point at infinity*. As we just saw, all rays with anisotropic "velocity" v converge to ∞. Similarly, all light rays (rays with isotropic "velocity" v) starting from the origin converge to $(0 : v : 0)$ in the third component, and those that start somewhere else land either in the second or third component.

The reason we introduced this conformal compactification is that G is supposed to act on all of it. This is the case because the standard representation of G on $\mathbb{R}^{p+1,q+1}$ preserves η, and hence the notion of isotropic vectors, and because it acts by means of injective maps.

Lemma 3.2.2. *The action of G (and even of K) on $\widehat{\mathbb{R}^{p,q}}$ is transitive.*

Proof. Write $q : \mathbb{R}^{p+1,q+1} \setminus \{0\} \to \mathbb{P}^{d+1}(\mathbb{R})$ for the projectivisation map. Let $q(u \oplus v) \in \widehat{\mathbb{R}^{p,q}}$ where $u \in \mathbb{R}^{p+1}, v \in \mathbb{R}^{q+1}$. That $u \oplus v$ is isotropic and non-null implies that $\|u\| = \|v\| > 0$ (Euclidean norms). Without loss of generality assume that $\|u\| = \|v\| = 1$. Since $SO(h)$ acts transitively on S^{h-1} for every $h > 1$, there are $m \in SO(p+1), n \in SO(q+1)$ with $me_1 = u, ne_{q+1} = v$. Then $m \oplus n \in SO(p + 1) \times SO(q + 1) = K \leq G$ with

$$(m \oplus n)(1 : 0 : 1) = q(me_1 \oplus ne_{q+1}) = q(u \oplus v),$$

where the 0 in $(1 : 0 : 1)$ is a tuple of d-many zeroes. Thus, everything is contained in the orbit of $(1 : 0 : 1)$. □

Corollary 3.2.3. *Let Q be the stabiliser of the origin $\iota(0)$, then $\widehat{\mathbb{R}^{p,q}} \cong G/Q$ as manifolds.*

We are now interested in what the Lie algebra $\mathfrak{q}$ looks like.

Lemma 3.2.4. *Let*

$$\Gamma = \{\epsilon_0, \epsilon_0 \pm \epsilon_i \mid 1 \leq i \leq q\} \cup \{\pm\epsilon_i, \epsilon_i \pm \epsilon_j, -\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq q\} \subseteq R,$$

then

$$\mathfrak{q} = \mathfrak{m}_{\mathfrak{p}} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha.$$

Proof. We have $g \in Q$ iff $g(e_0 + e_{d+1}) \in \mathbb{R}(e_0 + e_{d+1})$. Hence, $\xi \in \mathfrak{q}$ iff the same condition holds.

Let $1 \leq j \leq q$, then $D_i = F^{i,d+1-i}$ has only zeroes in the 0-th and $d+1$-st column, so $D_i \cdot (e_0 + e_{d+1}) = 0$, hence $D_1, \dots, D_q \in \mathfrak{q}$. Furthermore,

$$D_0(e_0 + e_{d+1}) = F^{0,d+1}(e_0 + e_{d+1}) = e_0 + e_{d+1},$$

so $D_0 \in \mathfrak{q}$ as well. I.e. $\mathfrak{a}_\mathfrak{p} \subseteq \mathfrak{q}$. Since $\mathfrak{q}$ is a Lie algebra, this means that $[\mathfrak{a}_\mathfrak{p}, \mathfrak{q}] \subseteq \mathfrak{q}$ and hence that

$$\mathfrak{q} = (\mathfrak{a}_\mathfrak{p} \oplus \mathfrak{m}_\mathfrak{p} \cap \mathfrak{q}) \oplus \bigoplus_{\alpha \in R} (\mathfrak{g}_\alpha \cap \mathfrak{q}).$$

So we only need to see what these intersections are.

Let $1 \leq j \leq q$ and $q < \mu \leq p$, then $F_{j\mu} \pm F_{d+1-j,\mu}$ only has nonzero entries only in the columns $j, d+1-j, \mu$, hence

$$(F^{j\mu} \pm F^{d+1-j,\mu})(e_0 + e_{d+1}) = 0 \in \mathbb{R}(e_0 + e_{d+1}),$$

whence $\mathfrak{g}_{\mp\epsilon_j} \subseteq \mathfrak{q}$.

Furthermore,

$$(F^{0\mu} \pm F^{d+1,\mu})(e_0 + e_{d+1}) = (F^{0\mu} \mp F^{\mu,d+1})(e_0 + e_{d+1}) = (-e_\mu \mp e_\mu),$$

hence $\mathfrak{g}_{\epsilon_0} \subseteq \mathfrak{q}$, but also $\mathfrak{g}_{-\epsilon_0} \cap \mathfrak{q} = 0$.

Let $1 \leq i \neq j \leq q$, then $F^{ij} \pm F^{i,d+1-j} + F^{d+1-i,j} \pm F^{d+1-i,d+1-j}$ and $F^{ij} \mp F^{i,d+1-j} - F^{d+1-i,j} \pm F^{d+1-i,d+1-j}$ have both only nonzero values in columns $i, j, d+1-i, d+1-j$. None of these are 0 or $d+1$, hence they both lie in $\mathfrak{q}$ as well. Consequently,

$$\mathfrak{g}_{-\epsilon_i \mp \epsilon_j}, \mathfrak{g}_{\epsilon_i \pm \epsilon_j} \subseteq \mathfrak{q}.$$

For $1 \leq j \leq q$ we have

$$(F^{0j} \pm F^{0,d+1-j} + F^{d+1,j} \pm F^{d+1,d+1-j})(e_0 + e_{d+1}) = -e_j \pm e_{d+1-j} - e_j \pm e_{d+1-j} \neq 0$$

and

$$(F^{0j} \mp F^{0,d+1-j} - F^{d+1,j} \pm F^{d+1,d+1-j})(e_0 + e_{d+1}) = -e_j \mp e_{d+1-j} + e_j \pm e_{d+1-j} = 0,$$

so $\mathfrak{g}_{-\epsilon_0 \mp \epsilon_i} \cap \mathfrak{q} = 0$ and $\mathfrak{g}_{\epsilon_0 \pm \epsilon_i} \subseteq \mathfrak{q}$.

Lastly, for $q < \mu, \mu \leq p$ the matrix $F^{\mu\nu}$ has only nonzero entries in columns μ, ν, which are both different from $0, d+1$, hence $\mathfrak{m}_\mathfrak{p} \subseteq \mathfrak{q}$. □

Corollary 3.2.5. *The subalgebra* $\mathfrak{q} \leq \mathfrak{g}$ *is maximal parabolic.*

Proof. If we pick

$$R^+ := \{\epsilon_i, \epsilon_i \pm \epsilon_j \mid 0 \leq i < j \leq q\},$$

this is a valid choice of positive roots, with simple roots

$$S := \{\epsilon_0 - \epsilon_1, \epsilon_2 - \epsilon_1, \dots, \epsilon_{q-1} - \epsilon_q, \epsilon_q\}$$

Picking $S' := S \setminus \{\epsilon_0 - \epsilon_1\}$, we have

$$\Gamma = R^+ \cup (R \cap \operatorname{span}(S')).$$

In particular, the minimal parabolic subalgebra associated to R^+ is contained in $\mathfrak{q}$, whence $\mathfrak{q}$ is parabolic.

For maximality, assume there is a parabolic subalgebra $\mathfrak{q} \leq \mathfrak{q}' \leq \mathfrak{g}$. According to [Kna96, proposition 7.76], $\mathfrak{q}'$ is associated to another subset S'' of primitive roots. Due to the inclusions, we'd have $S' \subseteq S'' \subseteq S$. Since S' and S differ only by one element, we have either $S' = S''$ or $S = S''$. In either case, one of the inclusions is an equality. □

Now, with this parabolic subgroup come a whole lot of subalgebras. In particular,

$$\begin{aligned}
\mathfrak{a} &= \bigcap_{i=1}^{q} \ker(\epsilon_i) = \mathbb{R}D_0 \\
\mathfrak{a}_M &= (\mathfrak{a})^{\perp} = \operatorname{span}\{D_1, \ldots, D_q\} \\
\mathfrak{m} &= \mathfrak{a}_M \oplus \mathfrak{m}_{\mathfrak{p}} \oplus \bigoplus_{\beta \in \Gamma \cap -\Gamma} \mathfrak{g}_\beta \\
&= \operatorname{span}\{F^{\mu\nu} \mid 1 \leq \mu, \nu \leq d\} \\
\mathfrak{n} &= \mathfrak{g}_{\epsilon_0} \oplus \bigoplus_{i=1}^{q} (\mathfrak{g}_{\epsilon_0+\epsilon_i} \oplus \mathfrak{g}_{\epsilon_0-\epsilon_i}) \\
&= \operatorname{span}\{K^\mu \mid 1 \leq \mu \leq d\} \\
\mathfrak{n}_M &= \bigoplus_{\alpha \in \Gamma \cap -\Gamma \cap R^+} \mathfrak{g}_\alpha \\
&= \operatorname{span}\left\{F^{\mu\nu} - F^{d+1-\mu,\nu} \,\middle|\, 1 \leq \mu \leq q, 1 \leq \nu \leq d\right\}
\end{aligned}$$

where $K^\mu := F^{0\mu} - F^{d+1,\mu}$. Analogously, define $P^\mu := -F^{0\mu} - F^{d+1,\mu}$ $(1 \leq \mu \leq d)$ and

$$\overline{\mathfrak{n}} = \operatorname{span}\{P^\mu \mid 1 \leq \mu \leq d\}.$$

Note that

$$-P_\mu = F_{0\mu} - F_{d+1,\mu}.$$

For $\Lambda^{\mu\nu}$ antisymmetric $(1 \leq \mu, \nu \leq d)$ and $b \in \mathbb{R}^{p,q}$ we then have

$$\begin{aligned}
\left[\frac{1}{2}\Lambda_{\mu\nu}F^{\mu\nu}, b_\rho K^\rho\right] &= \frac{1}{2}\Lambda_{\mu\nu}b_\rho\left[F^{\mu\nu}, F^{0\rho} - F^{d+1,\rho}\right] \\
&= \frac{1}{2}\Lambda_{\mu\nu}b_\rho\left(\eta^{\mu\rho}F^{\nu 0} - \eta^{\nu\rho}F^{\mu 0} - \eta^{\mu\rho}F^{\nu,d+1} + \eta^{\nu\rho}F^{\mu,d+1}\right) \\
&= -\Lambda_{\mu\nu}b_\rho\eta^{\nu\rho}\left(F^{\mu 0} - F^{\mu,d+1}\right) \\
&= \Lambda_{\mu\nu}b^\nu K^\mu \\
&= (\Lambda b)_\mu K^\mu,
\end{aligned}$$

and similarly for $x^\mu P_\mu$. As matrices we have

$$b_\mu K^\mu = \begin{pmatrix} 0 & b_\bullet^T & 0 \\ -b^\bullet & 0 & b^\bullet \\ 0 & -b_\bullet^T & 0 \end{pmatrix}, \qquad x_\mu P^\mu = \begin{pmatrix} 0 & -x_\bullet^T & 0 \\ x^\bullet & 0 & x^\bullet \\ 0 & -x_\bullet^T & 0 \end{pmatrix},$$

where $x_\bullet$ and $x^\bullet$ is the $d \times 1$-matrix in whose μ-th row is the entry x_μ or x^μ, respectively.

In order to show that Q is the parabolic subgroup associated to $\mathfrak{q}$. We need to recall some statements about parabolic subgroups.

Proposition 3.2.6. *Let A, N, A_M, N_M be the analytic subgroups of $\mathfrak{a}, \mathfrak{n}, \mathfrak{a}_M, \mathfrak{n}_M$, let $M = {}^0Z_G(\mathfrak{a})$, let $M_M = Z_{K\cap M}(\mathfrak{a}_M)$, and $K_M = K \cap M$. Then*

- *(a) $MA = Z_G(\mathfrak{a})$ is reductive, with A as its noncompact Abelian (vector space) component, and M as its remainder (also reductive).*
- *(b) M has Lie algebra $\mathfrak{m}$.*
- *(c) $M_M = M_\mathfrak{p}$, so that $M_\mathfrak{p} A_M N_M = M_M A_M N_M \leq M$ is a minimal parabolic subgroup. Furthermore, $K_M A_M N_M = M$ is an Iwasawa decomposition.*
- *(d) MA normalises N, so that $Q = MAN$ is a group.*
- *(e) $Q = N_G(\mathfrak{q}) \leq G$ is a closed subgroup.*
- *(f) Q has Lie algebra $\mathfrak{q}$.*
- *(g) Multiplication $M \times A \times N \to Q$ is a diffeomorphism.*
- *(h) $\overline{N} \cap Q = 1$.*
- *(i) $G = KQ$.*

Proof. [Kna96, proposition 7.82(a–c)] and [Kna96, proposition 7.83]. □

Lemma 3.2.7. *The group Q is the parabolic subgroup of the parabolic subalgebra $\mathfrak{q}$.*

Proof. Write Q' for the parabolic subgroup, then we need to show $Q = Q'$.
"$\supseteq$": We have the Langlands decomposition (Proposition 3.2.6(d)) $Q' = MAN$ where A, N are the analytic subgroups for $\mathfrak{a}, \mathfrak{n}$. Since $\mathfrak{a}, \mathfrak{n}$ fix $\iota(0)$, so do their analytic subgroups, so it remains to show that $M \subseteq Q'$. For that we use the Iwasawa decomposition of M, which is (Proposition 3.2.6(c)) $M = K_M A_M N_M$ with A_M, N_M the analytic subgroups for $\mathfrak{a}_M, \mathfrak{n}_M$ and $K_M = Z_K(\mathfrak{a})$. Since both $\mathfrak{a}_M, \mathfrak{n}_M$ fix $\iota(0)$, so do their analytic subgroups.

So it remains to check for K_M. Let

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in K$$

fix D_0 (write as block matrices with blocks $p+1, q+1$, both index from 1 to $p/q+1$). Written out this is

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & e_1 e_{q+1}^T \\ e_{q+1} e_1^T & 0 \end{pmatrix} \begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix} = \begin{pmatrix} 0 & A e_1 e_{q+1}^T B^T \\ B e_{q+1} e_1^T A^T & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & e_1 e_{q+1}^T \\ e_{q+1} e_1^T & \end{pmatrix},$$

hence $Ae_1(Be_{q+1})^T = e_1 e_{q+1}^T$, i.e. $Ae_1 = \lambda e_1$ and $Be_{q+1} = \lambda^{-1} e_{q+1}$ for $\lambda \in \mathbb{R}$. Since ± 1 are the only real eigenvalues an orthogonal matrix could have, we even have $\lambda^{-1} = \lambda$. Hence

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \iota(0) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot (e_1 : e_{q+1}) = (\lambda e_1 : \lambda e_{q+1}) = \iota(0),$$

hence K_M fixes $\iota(0)$.
"$\subseteq$": By Proposition 3.2.6(e), Q' is the normaliser of $\mathfrak{q}$ in G, so it suffices to show that Q normalises $\mathfrak{q}$. Let $\xi \in \mathfrak{q}$, and $g \in Q$, then

$$\exp(t \operatorname{Ad}(g)(\xi)) \cdot \iota(0) = g \exp(t\xi) g^{-1} \iota(0) = \iota(0),$$

for all t, hence $\operatorname{Ad}(g)(\xi) \in \mathfrak{q}$ as well. Thus, Q normalises $\mathfrak{q}$, and hence $Q \subseteq Q'$. □

3.3. Point Configurations

In order to substantiate the claim made before Example 1.2.2, we need to investigate the structure (as G-sets) of spaces of configurations of n points in $G/Q \cong \widehat{\mathbb{R}^{p,q}}$. From the example (and the subsequent ones), we can already see that we can't expect our n-point functions to be defined on all of $(G/Q)^n$, because we need all the x_{ij}^2 to be nonzero (among other things) for $\iota(x_1), \ldots, \iota(x_n)$. For general configurations of points we want the following:

Definition 3.3.1. *A tuple of n points $(x_1, \ldots, x_n)$ with $x \in \widehat{\mathbb{R}^{p,q}}$ is said to be* in general position *if $x_i = q(v_i)$ where the vectors $v_1, \ldots, v_n \in \mathbb{R}^{p+1,q+1}$ are linearly independent, but no pair of them is orthogonal. ($q : \mathfrak{R}^{p+1,q+1} \setminus \{0\} \to \mathbb{P}^{d+1}(\mathbb{R})$ is the projectivisation.)*
Write

$$\operatorname{GP}(\widehat{\mathbb{R}^{p,q}}, n) = \operatorname{GP}(G/Q, n)$$

for the set of n-tuples of points in general position.

The reason why we use this weird-looking condition of non-orthogonality is that it functions as a generalisation for "not being lightlike (isotropically) separated". In particular, we have

$$\eta\left(\begin{pmatrix}1-\eta(x,x)\\2x\\1+\eta(x,x)\end{pmatrix},\begin{pmatrix}1-\eta(y,y)\\2y\\1+\eta(y,y)\end{pmatrix}\right) = (1-\eta(x,x))(1-\eta(y,y)) + 4\eta(x,y) \tag{3.1}$$

$$\begin{aligned}&- (1+\eta(x,x))(1+\eta(y,y))\\&= 4\eta(x,y) - 2\eta(x,x) - 2\eta(y,y)\\&= -2\eta(x-y,x-y),\end{aligned} \tag{3.2}$$

so the inner product is nonzero precisely when $x-y$ is non-isotropic.

Corollary 3.3.2. *G acts naturally on* $\mathrm{GP}(G/Q,n)$.

Proof. G preserves the inner product on $\mathbb{R}^{p+1,q+1}$, hence the non-orthogonality is preserved. Similarly, G acts by means of injective maps, so linear independence is also preserved. □

Proposition 3.3.3. *The action of G on* $\mathrm{GP}(G/Q,2)$ *is transitive.*

Proof. Let $(x,y)\in \mathrm{GP}(G/Q,2)$. Pick $g\in G$ so that $g\cdot x=\infty$. Let $v\in\mathbb{R}^{p+1,q+1}$ be such that $g\cdot y = q(v)$. Then

$$v = \begin{pmatrix}v_0\\ \underline{v}\\ v_{d+1}\end{pmatrix}$$

is not orthogonal to $(1,0,-1)^T$. This means that

$$0\neq \eta(v,(1,0,-1)^T) = v_0+v_{d+1}.$$

Let $u := \frac{\underline{v}}{v_0+v_{d+1}}$, then (cf. proof of Proposition 3.2.1) $q(v)=\iota(u)$ and we have

$$\begin{aligned}\exp(-u\cdot P)\cdot q(v) &= q\left(\begin{pmatrix}1-\frac{u^2}{2} & u_\bullet^T & -\frac{u^2}{2}\\ -u^\bullet & 1 & -u^\bullet\\ \frac{u^2}{2} & -u_\bullet^T & 1+\frac{u^2}{2}\end{pmatrix}\begin{pmatrix}1-u^2\\2u^\bullet\\1+u^2\end{pmatrix}\right)\\ &= q\begin{pmatrix}1\\0\\1\end{pmatrix}\\ \exp(-u\cdot P)\cdot\infty &= q\left(\begin{pmatrix}1-\frac{u^2}{2} & u_\bullet^T & -\frac{u^2}{2}\\ -u^\bullet & 1 & -u^\bullet\\ \frac{u^2}{2} & -u_\bullet^T & 1+\frac{u^2}{2}\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right)\\ &= q\begin{pmatrix}1\\0\\-1\end{pmatrix}\end{aligned}$$

(interpreting the column vector u to equal the matrix $u^\bullet$), so that $\exp(-u\cdot P)g(x,y) = (\infty,\iota(0))$. □

Proposition 3.3.4. *The stabiliser of* $(\iota(0), \infty) \in \mathrm{GP}(G/Q, 2)$ *is* MA.

Proof. Write $H \leq G$ for the stabiliser we're looking for. Let $w \in G$ be such that $w \cdot \iota(0) = \infty$, then

$$H = \left\{ g \in G \;\middle|\; g \in Q, w^{-1}gw \in Q \right\} = Q \cap wQw^{-1}.$$

Pick $w \in K$ diagonal with ± 1 entries, say $w = \mathrm{diag}(w_0, \dots, w_{d+1})$, so that $w_0 = -w_{d+1}$. (If $q+1$ is odd and > 1, we can pick $A = r_i r_{d+1}$, the reflexion along e_i, for $p < i < d+1$; if $q = 0$, we can pick $w = r_0 r_d$.)

In any case we have $we_0 = \lambda e_0$ and $we_{d+1} = -\lambda e_{d+1}$, so that

$$\mathrm{Ad}(w)(D_0) = wD_0w = -D_0.$$

Furthermore, all D_i $(i > 0)$ are either being sign-flipped or not. The upshot is that $\mathrm{Ad}^*(w)(\Gamma) = -\Gamma$. As a consequence, $wNw = \overline{N}$. Furthermore, by Proposition 3.2.6(a), $MA = Z_G(\mathfrak{a})$, so that if $g \in MA$ we have

$$\mathrm{Ad}(wgw)(D_0) = -\mathrm{Ad}(wg)(D_0) = -\mathrm{Ad}(w)(D_0) = D_0,$$

so that $wgw \in MA$ as well. This shows that $wMAw = MA$. Consequently, $wQw = MA\overline{N}$. Evidently, we have $MA \subseteq H = MAN \cap MA\overline{N}$.

Let $g = man = m'a'\overline{n} \in H$, then

$$\overline{n} = a'^{-1}m'^{-1}man \in \overline{N} \cap Q,$$

which is trivial by Proposition 3.2.6(h). Thus, $H \subseteq MA$ as well. ☐

Corollary 3.3.5. *As* G*-sets we have* $\mathrm{GP}(G/Q, 2) \cong G/MA$.

Corollary 3.3.6. *Let* $n \geq 2$ *and* $(x_1, \dots, x_n) \in \mathrm{GP}(G/Q, n)$*, then there exists* $g \in G$ *such that*

$$g \cdot x_1 = \iota(0), \qquad g \cdot x_2 = \infty, \qquad g \cdot x_i = \iota(p_i) \quad (2 < i \leq n)$$

where $p_3, \dots, p_n \in \mathbb{R}^{p,q}$ *are linearly independent and no* $p_i - p_j$ *is isotropic (in other words: none of the* p_i *are lightlike separated), and also no* p_i *is isotropic (i.e. none of the* p_i *is lightlike).*

Proof. Since the property of being in general position extends to subsets, the points x_1, x_2 are in general position. Hence by Proposition 3.3.3, there is $g \in G$ such that $g \cdot x_1 = \iota(0), g \cdot x_2 = \infty$. Since all the $g \cdot x_i$ $(i > 3)$ are not lightlike separated from $g \cdot x_2 = \infty$, we can use the same argument as in the proof of Proposition 3.3.3 to see that $g \cdot x_3, \dots, g \cdot x_n$ are all contained in $\iota(\mathbb{R}^{p,q})$ and hence can be written uniquely as $g \cdot x_i = \iota(p_i)$.

By definition of general position we know that the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 - \eta(p_i, p_i) \\ 2p_i \\ 1 + \eta(p_i, p_i) \end{pmatrix}$$

are all linearly independent. This is equivalent to all p_i are linearly independent.

Lastly, we need those same vectors to all be pairwise non-orthogonal. Using (3.2) we see that this implies that all the $p_i - p_j$ are non-isotropic, as well as the $p_i = p_i - 0$. $\square$

Since this thesis will be almost exclusively concerned with 4-point functions, let's look at configurations of 4 points more closely.

Proposition 3.3.7. *Define* $u, v : \mathrm{GP}(G/Q, 4) \to \mathbb{R}$ *by*

$$u(q(v_1), \ldots, q(v_4)) := \frac{\eta(v_1, v_2)\eta(v_3, v_4)}{\eta(v_1, v_3)\eta(v_2, v_4)},$$
$$v(q(v_1), \ldots, q(v_4)) := \frac{\eta(v_1, v_4)\eta(v_2, v_3)}{\eta(v_1, v_3)\eta(v_2, v_4)}.$$

Both functions are well-defined and smooth, and for $\iota(x_1), \ldots, \iota(x_4)$ *they reduce to the well-known expressions for the cross-ratios from e.g. [PRV19, Section III.C.3].*

Proof. First-off note that the expressions defining u, v are invariant under independently rescaling v_1, v_2, v_3, v_4 because the RHS is a homogeneous function of degree 0 in all of them. Thus the function definition makes sense. That $(q(v_1), \cdots, q(v_4)) \in \mathrm{GP}(G/Q, 4)$ implies that $\eta(v_1, v_3)\eta(v_2, v_4) \neq 0$ (because neither v_1, v_3 nor v_2, v_4 are orthogonal). Thus, u, v are well-defined and smooth because q is a quotient map.

For the well-known expressions, recall (3.2). If we replace every occurrence of $\eta(v_i, v_j)$ in our definition with $-2\eta(x_i - x_j, x_i - x_j)$, all factors of -2 will cancel, and leave us with the familiar expression. $\square$

This way it becomes trivial to also evaluate the cross-ratios at infinity (without having to take a limit first):

$$\begin{aligned}
u(\iota(0), \infty, \iota(x), \iota(y)) &= \frac{-2\eta((1,0,1),(1,0,-1))\eta(x-y, x-y)}{-2\eta(x,x)\eta((1-\eta(y,y), 2y, (1+\eta(y,y))),(1,0,-1))} \\
&= \frac{-4\eta(x-y, x-y)}{-4\eta(x,x)} \\
&= \frac{\eta(x-y, x-y)}{\eta(x,x)} \\
v(\iota(0), \infty, \iota(x), \iota(y)) &= \frac{\eta(y,y)}{\eta(x,x)}.
\end{aligned}$$

And now for the all-important question, what G-orbits does $\mathrm{GP}(G/Q, 4)$ have? There are two ways of approaching this:
On one hand, we have

$$\mathrm{GP}(G/Q, 4) \subseteq \mathrm{GP}(G/Q, 2)^2 \cong (G/MA)^2$$

as G-sets, so that

$$\mathrm{GP}(G/Q, 4)/G \subseteq (\mathrm{GP}(G/Q, 2)^2)/G \cong (G/MA)^2/G \cong MA\backslash G/MA.$$

This means, we can view $\mathrm{GP}(G/Q,4)/G$ as isomorphic to $MA\backslash U/MA$, where $U \subseteq G$ is an open set that is invariant under left and right multiplication by MA.

On the other hand, we can proceed geometrically.

Lemma 3.3.8. *(a) Let $v_1, \ldots, v_d \in \mathbb{R}^{p,q}$ be positively oriented with $\eta(v_i, v_j) = \eta_{ij}$, then there is $m \in SO(p,q)$ with $me_i = v_i$.*

(b) Let $v_1, \ldots, v_d \in \mathbb{R}^{p,q}$ be positively oriented with $\eta(v_i, v_j) = \eta_{ij}$ for $i = 2, \ldots, d-1$ and

$$\eta(v_1, v_1) = 0, \eta(v_d, v_d) = 0, \eta(v_1, v_d) = -2,$$

then there is $m \in SO(p,q)$ such that $me_i = v_i$ for $2 \leq i \leq d-1$ and

$$m(e_1 + e_d) = v_1, \qquad m(-e_1 + e_d) = v_d.$$

Proof. (a) Take m to be the matrix whose columns are $v_1, \ldots, v_d$. Then positive orientation implies that the determinant is 1, and "orthonormality" implies that the matrix lies in $O(p,q)$.

(b) Apply the previous case to

$$\frac{v_1 - v_d}{2}, v_2, \ldots, v_{d-1}, \frac{v_1 + v_2}{2}.$$

This basis is positively oriented because the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

has determinant 2, and the inner products are correct. Then we have

$$m(\pm e_1 + e_d) = \pm\frac{v_1 - v_d}{2} + \frac{v_1 + v_d}{2} = v_{1/d},$$

which is what we wanted.

□

Proposition 3.3.9. *We have*

$$\mathrm{GP}(G/Q,4)/G = X_{tt} \sqcup X_{tl} \sqcup X_{ts} \sqcup X_{st} \sqcup X_{sl} \sqcup X_{ss},$$

where $X_{\sigma\tau}$'s principal domain is

$$\{(\iota(0), \infty, \iota(x), \iota(y)) \mid (x,y) \in Y_{\sigma\tau}\} \quad (\sigma \in \{s,t\}, \tau \in \{s,l,t\})$$

where

$$\begin{aligned}
Y_{tt} &= \{(e_d, ae_d + be_{d-1}) \mid b > 0\} \\
Y_{tl} &= \{(e_d, ae_d + e_1 + e_{d-1}) \mid a \neq 0, 1\} \\
Y_{ts} &= \{(e_d, ae_d + be_1) \mid b > 0, b \neq |a|, |a-1|\} \\
Y_{st} &= \{(e_1, ae_1 + be_d) \mid b > 0, b \neq |a|, |a-1|\} \\
Y_{sl} &= \{(e_1, ae_1 + e_2 + e_d) \mid a \neq 0, 1\} \\
Y_{ss} &= \{(e_1, ae_1 + be_2) \mid b > 0\}.
\end{aligned}$$

For $q \leq 1$ we take instead $Y_{tt} = Y_{tl} = \emptyset$, and for $q = 0$, we also take $Y_{ts} = Y_{st} = Y_{sl} = \emptyset$.

Proof. Let $(x_1, x_2, x_3, x_4) \in \mathrm{GP}(G/Q, 4)$. Since we're studying G-orbits, assume without loss of generality that

$$(x_1, x_2, x_3, x_4) = (\iota(0), \infty, \iota(x), \iota(y)),$$

which is possible by Corollary 3.3.6. Since by Proposition 3.3.4, the pair $(\iota(0), \infty)$ is stabilised by MA, we have to classify the orbits of

$$\{(x, y) \in \mathbb{R}^{p,q}, \text{linearly independent}, \eta(x, x), \eta(y, y), \eta(x - y, x - y) \neq 0\}$$

under MA, and these will turn out to be the Y's.

Note that

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \cdot \iota(x) = \iota(\pm m x)$$

for $m \in SO(p, q) \subseteq M$ (where, depending on the parity of p, q we may have to choose -1 instead of $+1$ if $m \notin SO(p, q)_0$), and that

$$\exp(\alpha D_0) \cdot \iota(x) = \begin{pmatrix} \cosh(\alpha) & 0 & \sinh(\alpha) \\ 0 & 1 & 0 \\ \sinh(\alpha) & 0 & \cosh(\alpha) \end{pmatrix} \cdot \iota(x) = \iota(\exp(-\alpha) x).$$

This means that what remains of our G-action can be used to rescale x, y (together) and to apply a transformation in $SO(p, q)$.

From Corollary 3.3.6 we know that x is not isotropic, so consider

$$\tilde{y} := y - \frac{\eta(x, y)}{\eta(x, x)} x.$$

Evidently, $\eta(x, \tilde{y}) = 0$. We now distinguish cases based on $\eta(x, x)$ and $\eta(\tilde{y}, \tilde{y})$.

x timelike, $\tilde{y}$ timelike We can find $v_1, \ldots, v_{d-2}$ so that

$$v_1, \ldots, v_{d-2}, \frac{\tilde{y}}{|\eta(\tilde{y}, \tilde{y})|}, \frac{x}{\sqrt{|\eta(x, x)|}}$$

is a positively oriented ONB, then by Lemma 3.3.8 there is $m \in SO(p, q)$ such that

$$m e_{d+1} = \frac{x}{\sqrt{|\eta(x, x)|}}, \qquad m e_d = \frac{\tilde{y}}{|\eta(\tilde{y}, \tilde{y})|}.$$

Multiplying by $\sqrt{|\eta(x, x)|}$ gives $ma \in MA$ such that

$$x = m a e_{d+1}, \qquad y \in m a \operatorname{span}\{e_d, e_{d+1}\},$$

where the coefficient in front of e_d is positive.

x **time/space-like,** $\tilde{y}$ **time/space-like** It works similarly in these cases, just tweaking the position inside the basis a bit.

x **timelike,** $\tilde{y}$ **lightlike** For this let $z \perp x$ be an isotropic vector that's not orthogonal to $\tilde{y}$ (e.g. take a timelike unit vector e_i with $\eta(\tilde{y}, e_i) \neq 0$, project it onto $(\mathbb{R}x)^\perp$, and take $\tilde{y}$'s reflexion along it). Then we can apply Lemma 3.3.8(b) to

$$\frac{\tilde{y}}{\sqrt{|\eta(x,x)|}}, \text{other vectors}, \frac{x}{\sqrt{|\eta(x,x)|}}, \frac{-2z}{\eta(z,\tilde{y})}\sqrt{|\eta(x,x)|}.$$

We see that there is $ma \in MA$ such that

$$ma \cdot (e_1 + e_d) = \tilde{y}, \qquad ma \cdot e_{d-1} = x.$$

Permuting around the unit vectors (all the while keeping our elements in MA), transforms out x, y to be contained in Y_{tl}.

x **spacelike,** $\tilde{y}$ **lightlike** Analogous.

□

We see that $\mathrm{GP}(G/Q, 4)/G$ is a union of 1- or 2-dimensional manifolds. In fact, considering the points $(e_1, e_2 + te_d)$, it is not hard to see that Y_{sl} lies in the closures of Y_{ss} and Y_{st} because $e_2 + te_d = \tilde{y}$ is spacelike for $t < 1$, lightlike for $t = 1$, and timelike for $t > 1$. Similarly, we can use $(e_d, e_1 + te_{d-1})$ to see that (for $q > 1$) Y_{tl} lies in the closures of Y_{ts} and Y_{tt}. And in fact, using the cross-ratios from Proposition 3.3.7, we can turn

$$Y_1 := X_{st} \cup X_{sl} \cup X_{ss}, \qquad Y_2 := X_{tt} \cup X_{tl} \cup X_{ts}$$

into smooth manifolds. Furthermore, the map

$$(\iota(0), \infty, \iota(x), \iota(y)) \mapsto \frac{\eta(x,x)}{|\eta(x,x)|}$$

induces a continuous map $\mathrm{GP}(G/Q, 4)/G \to \{\pm 1\}$, which assumes the value -1 on the first such manifold, and 1 on the other, so the subsets X_1, X_2 are in fact clopen. They both have four connected components each: the ones where $x-y, y$ are time- or spacelike.

This shows that $\mathrm{GP}(G/Q, 4)/G$ is a 2-dimensional manifold with eight connected components, corresponding to the sign of $\eta(x,x), \eta(x-y, x-y)$, and $\eta(y,y)$ (if $g = 1$, the connected component associated to $u, v > 0$ in the x timelike plane is smaller; and if $g = 0$, all vectors can only have positive inner product with themselves, so then only one connected component remains: only the Y_{ss} part of x spacelike, $u, v > 0$).

Proposition 3.3.10. *The permutation action of S_4 on $\mathrm{GP}(G/Q, 4)/G$, as described by two planes of cross-ratios, is as follows:*

$$(1234) \cdot (u, v) = (v, u), \qquad (12) \cdot (u, v) = \left(\frac{u}{v}, \frac{1}{v}\right)$$

where (1234) *maps the $v < 0$ components of X_i to X_{3-i} ($i = 1, 2$), and the $v > 0$ components of X_i to X_i. The cycle* (12) *leaves X_1, X_2 invariant.*

Proof. Determining the behaviour of (1234), (12) on the cross-ratios is a simple matter of evaluating u, v of

$$(\iota(y), \iota(0), \infty, \iota(x)), \qquad (\infty, \iota(0), \iota(x), \iota(y)).$$

Applying w to both sides gives us

$$(\iota(I(y)), \infty, \iota(0), \iota(I(x))), \qquad (\iota(0), \infty, \iota(I(x)), \iota(I(y)))$$

(cf. Section B.3 for $I(x)$). Translating by $-I(y)$ on the left gives us that the plane that $(1234) \cdot (\iota(0), \infty, \iota(x), \iota(y))$ resides in is determined by whether y is time- or spacelike. Similarly, the plane of $(12) \cdot (\iota(0), \infty, \iota(x), \iota(y))$ is determined whether or not x is time- or spacelike.

In particular, for $y > 0$, the lengths $\eta(x, x), \eta(y, y)$ have the same sign, so (1234) doesn't change between planes, and for $y < 0$, they have a different sign, making its action change the planes. (12), on the other hand, doesn't switch planes at all. □

Corollary 3.3.11. *The maps that move between the two pictures is given by*

$$\begin{aligned} MA(Y_1 \cup Y_2) \to U, &\qquad (x, y) \mapsto \exp(y \cdot P) w \exp(I(x - y) \cdot P) \\ \psi : U \mapsto \mathrm{GP}(G/Q, 4)/G, &\qquad g \mapsto G(MAN, wMAN, gMAN, gwMAN) \end{aligned}$$

Since the fundamental domains $Y_{\sigma\tau}$ are disconnected from each other, we shall tackle them separately. Since Y_{tl}, Y_{sl} are only one-dimensional and their orbits are contained in the closures of those of the others, we are going to ignore these domains for the time being. Define

$$\begin{aligned} \xi_{tt}(a, b) &:= \exp(P_d) w \exp(aP_d + bP_{d-1}) \\ \xi_{ts}(a, b) &:= \exp(P_d) w \exp(aP_d + bP_1) \\ \xi_{st}(a, b) &:= \exp(P_1) w \exp(aP_1 + bP_d) \\ \xi_{ss}(a, b) &:= \exp(P_1) w \exp(aP_1 + bP_2), \end{aligned}$$

then

$$\begin{aligned} u(\psi(\xi_{tt}(a, b))) &= \frac{1}{(a + w_0 w_d)^2 + b^2} & v(\psi(\xi_{tt}(a, b))) &= \frac{a^2 + b^2}{(a + w_0 w_d)^2 + b^2} \\ u(\psi(\xi_{ts}(a, b))) &= \frac{1}{(a + w_0 w_d)^2 - b^2} & v(\psi(\xi_{ts}(a, b))) &= \frac{a^2 - b^2}{(a + w_0 w_d)^2 - b^2} \\ u(\psi(\xi_{st}(a, b))) &= \frac{1}{(a - w_0 w_1)^2 - b^2} & v(\psi(\xi_{st}(a, b))) &= \frac{a^2 - b^2}{(a - w_0 w_1)^2 - b^2} \\ u(\psi(\xi_{ss}(a, b))) &= \frac{1}{(a - w_0 w_1)^2 + b^2} & v(\psi(\xi_{ss}(a, b))) &= \frac{a^2 + b^2}{(a - w_0 w_1)^2 + b^2} \end{aligned} \tag{3.3}$$

4. Induced Representations

In Section 1.2 we saw that the conformal Ward identities (that are satisfied by correlation functions and conformal blocks) can be phrased as co- and invariance conditions on functions or distributions defined on G^n (for n-point functions). In that context we also mentioned induced representations, which we shall have a closer look at now.

Definition 4.0.1. *Let (G, K, θ, B) be a reductive Lie group, let $MAN = Q \leq G$ be a parabolic subgroup. Let (π, V) be a finite-dimensional representation that is unitary when restricted to $K \cap M$ (with respect to the inner product $\langle\cdot\rangle$), and let $\nu \in \mathfrak{a}^*$.*

Extend (π, V) to a representation of Q by having

$$\pi(man)v = \pi(m)\exp(\nu(\log(a)))v.$$

Define

$$\mathrm{Ind}_Q^G(\pi) := \left\{ f : G \to V \,\middle|\, \forall g \in G, q \in Q : f(gq) = \pi(q)^{-1} f(g), f|_K \in L^2(K; V) \right\},$$

with $(g \cdot f)(x) := f(g^{-1}x)$.

Proposition 4.0.2. *The representation space $\mathrm{Ind}_Q^G(\pi)$ is a Hilbert space, and the action by G is strongly continuous.*

There are three equivalent ways of viewing $\mathrm{Ind}_Q^G(\pi)$: the definition is called the *induced picture*. In addition, there is also the *compact* and the *noncompact* picture.

Theorem 4.0.3 (Equivalence with Compact Picture)**.** *The map*

$$\mathrm{Ind}_Q^G(\pi) \to \left\{ f \in L^2(K; V) \,\middle|\, \forall g \in K, q \in K \cap M : f(gq) = \pi(q)^{-1} f(g) \right\}$$

defined by restriction is a unitary equivalence of G-modules.

Proof. G acts on f in the right set as

$$(g \cdot f)(k) = \exp(\nu(-\alpha))\pi(m')^{-1} f(k')$$

where $g^{-1}k = k'm' \exp(\alpha)n' \in G = KQ$ ($k' \in K, m' \in M, \alpha \in \mathfrak{a}, n' \in N$). Since f is equivariant with respect to $K \cap M$, the ambiguity in this decomposition doesn't matter. From this definition, it is clear that the restriction map intertwines G-representations.

For unitarity note that the L^2-inner product on K is also the inner product that we chose on $\mathrm{Ind}_Q^G(\pi)$. It remains to see surjectivity. Let $f \in L^2(K; V)$ be $K \cap M$-equivariant. Define $f' : G \to V$ by

$$f'(km\exp(\alpha)n) = \pi(m)^{-1}\exp(-\nu(\alpha))f(k).$$

This function is well-defined because of the equivariance of f, lies in $\mathrm{Ind}_Q^G(\pi)$, and restricts to f. ☐

Note that the representation space in the noncompact picture is independent of ν; only the representation depends on ν.

Theorem 4.0.4 (Equivalence with Non-Compact Picture)**.** *Let $H : G \to \mathfrak{a}$ be such that $\exp(H(kman)) = a$ for $k \in K, man \in MAN$ and note that*

$$\Delta_Q(q) = \exp(2\rho_A(H(q)))$$

([Kna96, equation (8.38)]) where ρ_A is half the sum of positive roots in the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Define

$$\delta : G \to \mathbb{R}, \qquad g \mapsto \exp(2(\operatorname{Re}(\nu) - \rho_A)(H(g))),$$

then restriction to $\overline{N}$ is a unitary equivalence of G-modules between

$$\operatorname{Ind}_Q^G(\pi) \to L^2(\delta \cdot \mu_{\overline{N}}; V).$$

Proof. We will focus on unitarity and then just define the G-action so that it intertwines.

Let $u, v \in \operatorname{Ind}_Q^G(\pi)$, define

$$f : G \to \mathbb{C}, \qquad g \mapsto \langle u(g), v(g)\rangle \delta(g)$$

(taking the inner product in V). This function satisfies

$$\begin{aligned} f(kman) &= \langle u(kman), v(kman)\rangle \exp(2(\operatorname{Re}(\nu) - \rho_A) \log(a)) \\ &= \langle \pi(man)^{-1} u(k), \pi(man)^{-1} v(k)\rangle \exp(2(\operatorname{Re}(\nu) - \rho_A) \log(a)) \\ &= \exp(-2 \operatorname{Re}(\nu) \log(a)) \langle u(k), v(k)\rangle \exp(2(\operatorname{Re}(\nu) - \rho_A) \log(a)) \\ &= \frac{1}{\Delta_Q(man)} \langle u(k), v(k)\rangle \\ &= \frac{f(k)}{\Delta_Q(man)}, \end{aligned}$$

so in particular, for $\overline{n} = kman$ we have

$$f(\overline{n}) = f(k) \exp(-2\rho_A(H(\overline{n}))).$$

In particular, this also shows that f is $K \cap M$-invariant.

By [Kna96, proposition 8.46], we thus obtain

$$\int f \,\mathrm{d}\mu_K = \int f \,\mathrm{d}\mu_{\overline{N}},$$

and hence that the inner products of u, v coincide, whether they are taken over K with μ_K or over $\overline{N}$ with $\delta \cdot \mu_{\overline{N}}$.

For surjectivity let $f \in L^2(\delta \cdot \mu_{\overline{N}}; V)$ and define

$$f'(\overline{n}man) := \pi(man)^{-1} f(\overline{n})$$

on $\overline{N}Q$ and 0 everywhere else. Then f' is equivariant almost everywhere and its K-norm is finite because of the equality of inner products. Consequently, $f' \in \operatorname{Ind}_Q^G(\pi)$ and restricts to f. ☐

Instead of requiring $\mathrm{Ind}_Q^G(\pi)$ to be made up of L^2-functions, we can also require its elements to be smooth functions $G \to V$ or V-valued distributions on G that satisfy the equivariance condition. It is sound to still refer to these different vector spaces as "the same representation" because if we consider their sets of K-finite vectors (their *Harish-Chandra modules*)

$$\left\{ f \in \mathrm{Ind}_Q^G(\pi) \,\middle|\, \dim(\mathrm{span}(K \cdot f)) < \infty \right\},$$

they are isomorphic as $(U(L), K)$-modules ($L = \mathfrak{g} \otimes \mathbb{C}$).

Considering smooth functions or derivations instead of L^2-functions means that our representation spaces are no longer Hilbert spaces but complete locally convex topological vector spaces. However, it turns out (see Appendix A.3) that we have more than that because they are also nuclear spaces. As such, there is a canonical notion of complete(d) tensor product between them and we have

Proposition 4.0.5. *For semisimple Q-modules $(V_1, \pi_1), \dots, (V_n, \pi_n)$ we have*

$$\mathrm{Ind}_Q^G(\pi_1) \otimes \cdots \otimes \mathrm{Ind}_Q^G(\pi_n) \cong \mathrm{Ind}_{Q^n}^{G^n}(\pi_1 \otimes \pi_n)$$

where we Ind *can refer to the Hilbert space, the smooth, and the distributional version.*

Proof. Note that $(G^n, K^n, \theta^{\otimes n}, B^{\otimes n})$ is a reductive group, with $Q^n \leq G^n$ parabolic. Similarly, $\pi_1 \otimes \cdots \otimes \pi_n$ is a semisimple Q^n-module.

Next, we have $L^2(G; V)^{\otimes n} \cong L^2(G^n; V^{\otimes n})$ as well as $C^\infty(G; V)^{\otimes n} \cong C^\infty(V^n; V^{\otimes n})$ and

$$(\mathcal{D}(G; V)')^{\otimes n} \cong \mathcal{D}(G^n; V^{\otimes n})',$$

and the equivariance condition survives taking the tensor product. □

4.1. n-Point Functions

Back to our concrete choice of G. As established in the introduction, n-point functions and conformal blocks satisfy the conformal Ward identities and are therefore G-invariant elements of $\mathrm{Ind}_{Q^n}^{G^n}(\pi^{\otimes n})$ (distributions). We expect them to be regular (i.e. a smooth functions) on $\mathrm{GP}(G/Q, n)$.

Let $G \times Q^n$ act on G^n by

$$(g, q_1, \dots, q_n) \cdot (g_1, \dots, g_n) := (g g_1 q_1^{-1}, \dots, g g_n q_n^{-1})$$

and on $V_1 \otimes \cdots \otimes V_n$ or $V^{\otimes n}$ by

$$(g, q_1, \dots, q_n) \cdot (v_1 \otimes \cdots \otimes v_n) := (q_1 \cdot v_1) \otimes \cdots \otimes (q_n \otimes v_n).$$

Definition 4.1.1. *Let $U_n \subseteq G^n$ be such that $U_n / Q^n \cong \mathrm{GP}(G/Q, n)$. It, too, is evidently a $G \times Q^n$-set.*

In this language, our solutions (in n variables) of the conformal Ward identities are functions $f : U_n \to V_1 \otimes \cdots \otimes V_n$ such that

$$f(g \cdot p) = g \cdot f(p)$$

for all $p \in U_n$ and $g \in G \times Q^n$. This reads a lot like the kind of covariance condition satisfied by sections of a vector bundle associated to the "principal bundle" $U_n \to \mathrm{GP}(G/Q, n)/G$. Problem is that $U_n \to \mathrm{GP}(G/Q, n)/G$ is not a principal bundle because the action has stabilisers:

Lemma 4.1.2. *For $n > 2$, the $G \times Q^n$-set U_n has stabiliser algebras isomorphic to $\mathfrak{so}_{\mathbb{C}}(d + 2 - n)$.*

Proof. Evidently, the action of Q^n on U_n is free, so it remains to find the stabiliser of any element of $\mathrm{GP}(G/Q, n)$. By Corollary 3.3.6, any element of $\mathrm{GP}(G/Q, n)$ lies in the same orbit as $(\iota(0), \infty, \iota(x_3), \ldots, \iota(x_n))$. The points $\iota(0), \infty$ are stabilised by MA by Proposition 3.3.4. Thus, our stabiliser is the subgroup of MA stabilising $x_3, \ldots, x_n$. Since A scales the vectors, no element of A stabilises $x_3, \ldots, x_n$. Let $m \in M$ be contained in the stabiliser. Since $x_3, \ldots, x_n$ are all non-null and linearly independent, we can do Gram–Schmidt orthogonalisation and find $v_3, \ldots, v_n \in \operatorname{span}\{x_3, \ldots, x_n\}$ that are an orthonormal basis (with $\eta(v_i, v_i) = \pm 1$, as we're working in indefinite signature generally). Thus, there exists $g \in M$ such that $gv_i = e_{m_i}$, so that

$$\mathrm{Stab}(x_3, \ldots, x_n) = \mathrm{Stab}(v_3, \ldots, v_n) = g^{-1}\, \mathrm{Stab}(e_{m_3}, \ldots, e_{m_n})g.$$

The middle stabiliser group has complexified Lie algebra $\mathfrak{so}_{\mathbb{C}}(d+2-n)$, as fixing $n-2$ unit vectors implies that we're effectively dealing with $(d + 2 - n) \times (d + 2 - n)$-matrices. ☐

This means, when talking about $G \times Q^n$-equivariant functions $U_n \to V_1 \otimes V_n$, we have to be wary of nontrivial stabilisers. If we still want to phrase this in terms of vector bundles, there is no general overall mechanism we can use like in the case of free actions. Because of that we're now making an assumption:

Assumption 4.1.3. *The set*

$$\tilde{E}(V_1, \ldots, V_n) := \left\{ (x, v) \;\middle|\; x \in U_n, v \in (V_1 \otimes \cdots \otimes V_n)^{\mathrm{Stab}(x)} \right\}$$

is a manifold and a smooth $G \times Q^n$-set (combining the actions on Q_n and the tensor product of the V_i) whose orbit space $E(V_1, \ldots, V_n)$ is also manifold.

Furthermore, if $\phi_i : V_i \to W_i$ are Q-intertwiners, the induced map

$$\tilde{E}(V_1, \ldots, V_n) \to \tilde{E}(W_1, \ldots, W_n), (x, v) \mapsto (x, (\phi_1 \otimes \cdots \otimes \phi_n)(v))$$

is smooth and descends to a smooth map $\phi : E(V_1, \ldots, V_n) \to E(W_1, \ldots, W_n)$. If all ϕ_i are injective, ϕ is an immersion, if all ϕ_i are surjective, ϕ is a submersion.

If it turns out to be false, it doesn't change the gist of the statements that rely on it and can likely be remedied by removing some closed submanifolds of lower dimension.

Theorem 4.1.4. *The solutions (in n variables) to the conformal Ward identities are precisely the sections of a vector bundle over* $\mathrm{GP}(G/Q,n)/G$ *with typical fibre* $(V^{\otimes n})^{\mathfrak{so}_{\mathbb{C}}(d+2-n)}$.

Proof. Under Assumption 4.1.3, both $E(V,\ldots,V)$ and $E(0,\ldots,0)$ are manifolds, the latter being $\mathrm{GP}(G/Q,n)/G$. The projection map $G\times Q^n(x,v)\mapsto G\times Q^n\cdot x$ descends to a smooth surjective submersion $E(V,\ldots,V)\to\mathrm{GP}(G/Q,n)/G$ whose fibres are isomorphic to $(V^{\otimes n})^{\mathrm{Stab}(x)}$ for any $x\in U_n$. Furthermore, since the projector from the trivial bundle $\underline{V^{\otimes n}}$ to $E(V,\ldots,V)$ is smooth, $E(V,\ldots,V)$ is a vector bundle.

The sections of $E(V,\ldots,V)\to\mathrm{GP}(G/Q,n)/Q$ correspond to smooth functions $g_n: U_n\to V^{\otimes n}$ satisfying

$$\begin{aligned} f(gg_1p_1^{-1},\ldots,gg_np_n^{-1}) &= f((g,p_1,\ldots,p_n)\cdot(g_1,\ldots,g_n)) \\ &= (p_1,\ldots,p_n)f(g_1,\ldots,g_n) \end{aligned}$$

for all $(g,p_1,\ldots,p_n)\in G\times Q^n$ and $(g_1,\ldots,g_n)\in U_n$. That's precisely the requirement of Equations 1.3 and 1.4. □

Corollary 4.1.5. *The space of* 2*-point solutions to the Ward identities for the irreducible* Q*-modules* V_1,V_2 *is one-dimensional precisely when* V_1,V_2 *have related* M*-representations and the same scaling dimension.*

The space of 3-point solutions to the Ward identities for the irreducible Q*-modules* V_1,V_2,V_3 *is also finite-dimensional.*

Proof. For $n=2$, the set $\mathrm{GP}(G/Q,2)/G$ comprises just one point, so pick the singleton $(\iota(0),\infty)$ as fundamental domain. By Proposition 3.3.4, its stabiliser is

$$\left\{(ma,ma,c_w(m)a^{-1})\;\middle|\;ma\in MA\right\}.$$

This group has a nontrivial set of invariants in $V_1\otimes V_2$ iff V_1,V_2 have the same scaling dimension and if $(V_1\otimes\widetilde{V_2})^M$ is nontrivial ($\tilde{V_2}$ has the w-conjugated action). In particular, since V_1,V_2 are finite-dimensional irreducible $\mathfrak{m}$-modules, so by Schur's lemma, we have

$$(V_1\otimes\widetilde{V_2})^{\mathfrak{m}}\cong\begin{cases}\mathbb{C} & V_1^*\cong\widetilde{V_2}\\ 0 & \text{otherwise}\end{cases}.$$

For $n=3$, the set $\mathrm{GP}(G/Q,3)/G$ comprises either one or two points: by Corollary 3.3.6, any orbit contains an element of the shape $(\iota(0),\infty,\iota(x))$ with $x^2\neq0$. Then x can be rotated and rescaled to be a standard unit vector of length squared either 1 or -1. Consequently, if $q=0$, the orbit space comprises one point; if $q=1$, it comprises two points. Thus, the space of sections is either

$$(V_1\otimes V_2\otimes V_3)^{SO(p-1)}$$

(for $q=0$) or

$$(V_1\otimes V_2\otimes V_3)^{SO(p-1,q)}\oplus(V_1\otimes V_2\otimes V_3)^{SO(p,q-1)}.$$

□

Corollary 4.1.6. *The space of 4-point solutions to the Ward identities is the space of sections of a vector bundle of typical fibre* $(V^{\otimes 4})^{\mathfrak{so}_{\mathbb{C}}(d-2)}$ *over a two-dimensional manifold.*

Proof. By Proposition 3.3.9, the manifold $\mathrm{GP}(G/Q,4)/G$ is two-dimensional. $\square$

Lemma 4.1.7. *The space of 4-point solutions to the Ward identities with the reps* $(\pi_1, V_1), \ldots, (\pi_4, V_4)$ *is in linear bijection to*

$$\mathrm{Ward}(W) := \{f \in C^\infty(U; W) \mid \forall x \in U, p, q \in MA : f(pxq) = p \cdot f(x) \cdot q\}$$

where $W = V_1 \otimes \cdots \otimes V_4$ *is an* MA*-bimodule via*

$$p \cdot v \cdot q := \pi_1(p) \otimes \pi_2(c_w(p)) \otimes \pi_3(q^{-1}) \otimes \pi_4(c_w(q^{-1}))v.$$

Recall that $U \subseteq G$ was defined before Lemma 3.3.8 to consist of those elements $g \in G$ such that

$$\begin{aligned} MAgMA \approx G(MA, gMA) \in (G/MA)^2/G \approx G(Q, wQ, gQ, gwQ) \in \mathrm{GP}(G/Q,2)^2/G \\ = G(\iota(0), \infty, g \cdot \iota(0), g \cdot \infty) \end{aligned}$$

is an orbit of configurations in general position.

Proof. Define

$$\chi : \{\text{sol. to Ward identities}\} \to \mathrm{Ward}(W), \qquad f \mapsto (g \mapsto f(1, w, g, gw)).$$

This is well-defined because for $p, q \in MA$ we have

$$\begin{aligned} \chi(f)(pgq) &= f(1, w, pgq, pgqw) = f(p^{-1}, p^{-1}w, gq, gqw) \\ &= f(p^{-1}, wc_w(p^{-1}), gq, gwc_w(q^{-1})) \\ &= \pi_1(p) \otimes \pi_2(c_w(p)) \otimes \pi_3(q^{-1}) \otimes \pi_4(c_w(q^{-1}))f(1, w, g, gw) \\ &= p \cdot \chi(f)(g) \cdot q. \end{aligned}$$

For injectivity let $\chi(f) = 0$, then $f(1, w, g, gw) = 0$ for all $g \in G$. Let $p \in U_4$. By Corollary 3.3.6, there is $g \in G$ such that $(g \cdot p)/Q^4 = (\iota(0), \infty, \iota(x), \iota(y))$. Furthermore, since $(\iota(x), \iota(y)) \in \mathrm{GP}(G/Q, 2)$, there is $h \in G$ such that $h(\iota(x), \iota(y)) = (\iota(0), \infty)$. This shows that

$$gpQ^4 = (Q, wQ, h^{-1}Q, h^{-1}wQ)$$

or that

$$p = (g^{-1}q_1, g^{-1}wq_2, g^{-1}h^{-1}q_3, g^{-1}h^{-1}wq_4)$$

for $q_1, \ldots, q_4 \in Q$, so that

$$\begin{aligned} f(p) &= f(g^{-1}q_1, g^{-1}wq_2, g^{-1}h^{-1}q_3, g^{-1}h^{-1}wq_4) \\ &= f(q_1, wq_2, h^{-1}q_3, h^{-1}wq4) \\ &= \pi_1(q_1^{-1}) \otimes \pi_2(q_2^{-1}) \otimes \pi_3(q_3^{-1}) \otimes \pi_4(q_4^{-1})f(1, w, h^{-1}, h^{-1}w) \\ &= 0. \end{aligned}$$

For surjectivity let $h \in \mathrm{Ward}(W)$, let $(g_1, \dots, g_4) \in U_4$, and let $g, g' \in G, p_1, p_2, p_3, p_4 \in Q$ such that $(g_1, g_2) = (gp_1, gwp_2)$ and

$$g^{-1}g_3 = g'p_3, \qquad g^{-1}g_4 = g'wp_4.$$

Then define

$$f(g_1, g_2, g_3, g_4) := \pi_1(p_1^{-1}) \otimes \cdots \otimes \pi_4(p_4^{-1})h(g').$$

This is well-defined because the ambiguities in the choice of g, g' are resolved by h's biequivariance. Then $\chi(f) = h$. □

Lemma 4.1.8. *A function $f \in \mathrm{Ward}(V)$ is uniquely determined by the four functions $f \circ \xi_{\sigma\tau}$ for $\sigma, \tau \in \{s, t\}$.*

Proof. By Proposition 3.3.9 and some considerations afterwards, the set $X_{ss} \cup X_{st} \cup X_{ts} \cup X_{tt}$ is dense in $\mathrm{GP}(G/Q, 4)/G$. Since $\psi(\mathrm{im}(\xi_{\sigma\tau}))$ is a G-fundamental domain for $X_{\sigma\tau}$, we can apply ψ^{-1} and obtain that the closure of the $MA \times MA$-orbit of the union of the images of the $\xi_{\sigma\tau}$ is all of U. Consequently, f's continuity and biequivariance ensure that f is determined by its values on these four fundamental domains. □

Therefore, for $f \in \mathrm{Ward}(V)$ define

$$f_{\sigma\tau} := f \circ \xi_{\sigma\tau} \qquad \sigma, \tau \in \{s, t\}\,.$$

Proposition 4.1.9. *Let $\sigma, \tau \in \{s, t\}$. Then the image of $\xi_{\sigma\tau}$ has the common stabiliser*

$$\mathrm{Stab}(\xi_{\sigma\tau}(a, b)) \cong \begin{cases} SO(p, q-2) & \sigma\tau = tt \\ SO(p-1, q-1) & \sigma\tau \in \{st, ts\} \\ SO(p-2, q) & \sigma\tau = ss \end{cases}$$

(where the isomorphism is effected by embedding into MA as "obvious" block matrices in M, and then by embedding into $MA \times MA$ via $p \mapsto (p, c_w(p))$).

Proof. The image of $\xi_{\sigma\tau}$ is the image of $Y_{\sigma\tau}$ under the first isomorphism in Corollary 3.3.11. Therefore,

$$(g, g') \in \mathrm{Stab} \Leftrightarrow g' = c_w(g) \& g \in \mathrm{Stab}(Y_{\sigma\tau}),$$

implying that $g \in MA$ such that g fixes the two standard unit vectors used in the definition of $Y_{\sigma\tau}$. □

This shows that up to boundary conditions for $Y_{\sigma l}$, the restriction $f_{\sigma\tau}$ can vary freely in the vector space

$$V^{\mathrm{Stab}(Y_{\sigma\tau})}.$$

As (3.3) establishes, we can also use u, v instead of a, b to parametrise $f_{\sigma\tau}$, which is what we will do from now on.

4.2. Conformal Blocks

With this last equivalence established, we will refer to $\mathrm{Ward}(V)$ as the space of *solutions to the conformal Ward identities*, with respect to the MA-bimodule V (regardless of whether V came about in the manner described in Lemma 4.1.7). Now we'd like to find out how to find conformal blocks within $\mathrm{Ward}(V)$. In Section 1.6, we introduced conformal blocks as solutions to both the conformal Ward identities 1.3 and 1.4 and the Casimir equation (1.8). In other words: conformal blocks are elements of $\mathrm{Ward}(V)$ that diagonalise a particular action of $Z(U(L))$, where $L = \mathfrak{g} \otimes \mathbb{C}$.

Lemma 4.2.1. *Let V be an MA-bimodule, then $C^\infty(U; V)$ is a $(U(L), MA \times MA)$-module, where $MA \times MA$ acts on $U(L)$ by*

$$(p, p') \cdot q := \mathrm{Ad}(p)(q),$$

where Ad *is the continuation of the adjoint representation to all of $U(L)$.*

Proof. Let $\mathfrak{g}$ act by

$$(\xi \cdot f)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\, f(\exp(-t\xi)x)\bigg|_{t=0},$$

i.e. by right-invariant vector fields, and $MA \times MA$ by

$$((p, p') \cdot f)(x) := p \cdot f(p^{-1}xp') \cdot p'^{-1}.$$

Let $p, p' \in MA$ and $\xi \in \mathfrak{g}$, then

$$\begin{aligned}
((p, p') \cdot (\xi \cdot ((p, p')^{-1} \cdot f)))(x) &= p \cdot (\xi \cdot ((p, p')^{-1} \cdot f))(p^{-1}xp') \cdot p'^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, p \cdot ((p, p')^{-1} \cdot f)(\exp(-t\xi)p^{-1}xp') \cdot p'^{-1}\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, f(p \exp(-t\xi)p^{-1}x)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, f(\exp(-t\,\mathrm{Ad}(p)(\xi))x)\bigg|_{t=0} \\
&= (\mathrm{Ad}(p)(\xi) \cdot f)(x),
\end{aligned}$$

so the actions of $\mathfrak{g}$ and $MA \times MA$ are indeed compatible in the way claimed. Through complexification and the universal property of $U(L)$, this then also holds for $U(L)$ and $MA \times MA$. □

Corollary 4.2.2. $\mathrm{Ward}(V)$ *is a $U(L)^{MA}$-module, in particular a $Z(U(L))$-module.*

Proof. As $\mathrm{Ward}(V)$ is the space of $MA \times MA$-invariants of the $(U(L), MA \times MA)$-module $C^\infty(U; V)$, we can apply Lemma 2.1.3 and obtain the claim. □

It turns out, in the situation of Lemma 4.1.7, i.e. $V = V_1 \otimes \cdots \otimes V_4$ as vector spaces, this $Z(U(L))$ action is precisely the one used in (1.8).

Proof. Define the function

$$\tilde{\chi} : C^\infty(U_4; V) \to C^\infty(U; V), \qquad f \mapsto (g \mapsto f(1, w, g, gw)),$$

which becomes the function χ from the proof of Lemma 4.1.7 when acting on $G \times Q^4$-invariants. Then it suffices to show that it intertwines the $\mathfrak{g}$-actions

$$(\xi \cdot f)(g_1, \dots, g_4) = \frac{\mathrm{d}}{\mathrm{d}t} f(g_1, g_2, \exp(-t\xi)g_3, \exp(-t\xi)g_4)\Big|_{t=0}$$

on $C^\infty(U_4; V)$ and

$$(\xi \cdot f)(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(-t\xi)x)\Big|_{t=0}$$

on $C^\infty(U; V)$. Then it also intertwines the L and $U(L)$ and $Z(U(L))$-actions, in particular when restricted to $G \times Q^4$-invariant functions.

In order to show this intertwining property, note that

$$\begin{aligned}
(\xi \cdot \tilde{\chi}(f))(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\chi}(f)(\exp(-t\xi)x) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(1, w, \exp(-t\xi)x, \exp(-t\xi)xw) \\
&= (\xi \cdot f)(1, w, x, xw) \\
&= \chi(\xi \cdot f).
\end{aligned}$$

□

Definition 4.2.3. *Let χ be the central character of a $U(L)$-module and let V be an MA-bimodule. The solution $f \in \mathrm{Ward}(V)$ to the conformal Ward identities is called a* χ-conformal block *(with respect to V) if it satisfies the Casimir equation*

$$\forall z \in Z(U(L)) : z \cdot f = \chi(z) f.$$

4.3. Spinorial Conformal Blocks

The Casimir equation is a system of differential equations of order at least two (as $\mathfrak{so}(d+2, \mathbb{C}) = L$ is simple), and the differential operator associated to the quadratic Casimir element can be directly associated to the Laplacian on $(G/Q)^2$ (i.e. on two factors of $\mathrm{GP}(G/Q, 4)$) with respect to a G-invariant metric.

This might lead one to wonder if there is also a way of having "the Dirac operator" act on solutions to the conformal Ward identities. In particular, for the operator, we will be following the approach of [Kos99].

Proposition 4.3.1. *The map $\tilde{\theta} : \mathfrak{g} \to \mathfrak{g}$ defined by $x \mapsto \tilde{\eta} x \tilde{\eta}$ for*

$$\tilde{\eta} = \mathrm{diag}(-1, 1, \dots, 1, -1)$$

is an involutive Lie algebra automorphism making $(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{g})$ into a symmetric pair.

Proof. The diagonal matrices $\eta, \tilde{\eta}$ commute, hence $\tilde{\theta}$ indeed maps $\mathfrak{g} \to \mathfrak{g}$. Since $\tilde{\eta}^2 = 1$, our map is an involution and

$$\tilde{\theta}([x,y]) = \tilde{\eta}x\tilde{\eta}\tilde{\eta}y\tilde{\eta} - \tilde{\eta}y\tilde{\eta}\tilde{\eta}x\tilde{\eta} = \left[\tilde{\theta}(x), \tilde{\theta}(y)\right]$$

for $x, y \in \mathfrak{g}$, hence $\tilde{\theta}$ is also a Lie algebra automorphism.

Now, let's determine the eigenspaces of $\tilde{\theta}$. We have

$$\tilde{\theta}(F_{\mu\nu}) = \tilde{\eta}F_{\mu\nu}\tilde{\eta} = \begin{cases} F_{\mu\nu} & \mu, \nu \in \{0, d+1\} \\ -F_{\mu\nu} & \mu \in \{0, d+1\} \,\&\nu \in \{1, \ldots, d\} \text{ or vice-versa} \end{cases}.$$

In other words: the 1-eigenspace consists of $\mathfrak{m} \oplus \mathfrak{a}$ and the -1-eigenspace consists of $\mathfrak{n} \oplus \bar{\mathfrak{n}}$. □

In [Kos99, section 1], the author discusses so-called *representations of Lie type* as natural settings for his Dirac operator. These are orthogonal representations Y (there called $\mathfrak{p}$) of a Lie algebra $\mathfrak{r}$, equipped with a way to make $\mathfrak{r} \oplus Y$ into a Lie algebra, where $\mathfrak{r}$ is a subalgebra that normalises Y (the adjoint action is given by the representation). These requirements already specify

$$[\cdot,\cdot]|_{\mathfrak{r}\times\mathfrak{r}}, \qquad [\cdot,\cdot]|_{\mathfrak{r}\times Y}, [\cdot,\cdot]|_{Y\times\mathfrak{r}},$$

so it remains to specify the restriction to $Y \times Y$. In [Kos99, theorem 1.50], it is pointed out that the restriction to $Y \times Y$ is uniquely specified by an element of $(\bigwedge^3 Y)^{\mathfrak{r}}$ that satisfies some extra conditions. We take $\mathfrak{r} := \mathfrak{m} \oplus \mathfrak{a}$ and $Y := \mathfrak{n} \oplus \bar{\mathfrak{n}}$. They form a symmetric pair, and are hence of Lie type ([Kos99, theorem 1.59]). If, however, we forgot for a second about the Lie structure we already know to exist on $\mathfrak{r} \oplus Y$, we can note that

Proposition 4.3.2. $(\bigwedge^3 Y)^{\mathfrak{r}} = 0$.

Proof. Since the tuple $P_1, \ldots, P_d, K_1, \ldots, K_d$ is a basis for Y, a basis for $\bigwedge^3 Y$ is given by

$$P_\mu \wedge P_\nu \wedge P_\rho, \qquad P_\mu \wedge P_\nu \wedge K_\rho, \qquad P_\mu \wedge K_\nu \wedge K_\rho, \qquad K_\mu \wedge K_\nu \wedge K_\rho$$

where indices belonging to the same letter (K or P) are assumed to be strictly decreasing. Since the P_i, K_i also parametrise the eigenspaces of $\mathrm{ad}(D_0)$ in Y, we expect the same in $\bigwedge^3 Y$. Indeed, we have

$$\mathrm{ad}(D_0)(P_\mu \wedge P_\nu \wedge P_\rho) = -3P_\mu \wedge P_\nu \wedge P_\rho \qquad \mathrm{ad}(D_0)(P_\mu \wedge P_\nu \wedge K_\rho) = -P_\mu \wedge P_\nu \wedge K_\rho$$
$$\mathrm{ad}(D_0)(P_\mu \wedge K_\nu \wedge K_\rho) = P_\mu \wedge K_\nu \wedge K_\rho \qquad \mathrm{ad}(D_0)(K_\mu \wedge K_\nu \wedge K_\rho) = 3K_\mu \wedge K_\nu \wedge K_\rho.$$

We see that our basis indeed diagonalises $\mathrm{ad}(D_0)$, and that 0 is not an eigenvalue of $\mathrm{ad}(D_0)$. Thus, the only element of $\bigwedge^3 Y$ annihilated by $\mathrm{ad}(D_0)$, and hence by $\mathfrak{r}$, is 0. □

In conclusion, there exists no other Lie algebra structure on $\mathfrak{r} \oplus Y$ that we could be using here.

We now define $\mathcal{A} := \mathrm{C}\ell(Y) \otimes U(\mathfrak{g})$ and consider the following embeddings of $\mathfrak{r}$:

$$\begin{aligned} \alpha_Y &: \mathfrak{r} \xrightarrow{\mathrm{ad}} \mathfrak{so}(Y) \xrightarrow{j} \mathrm{C}\ell(Y) \\ \Delta_Y &: \mathfrak{r} \ni \xi \mapsto 1 \otimes \xi + \alpha_Y(\xi) \otimes 1 \in \mathcal{A}, \end{aligned} \tag{4.1}$$

where j is the Chevalley embedding from Proposition 2.2.11. If $Y_1, \dots, Y_{2d}$ is an orthonormal basis of Y, the map α_Y can be given by

$$\xi \mapsto \frac{1}{4} \sum_{i,j=1}^{2d} B(\xi, [Y_i, Y_j]) Y_i Y_j.$$

Definition 4.3.3. *Let $Y_1, \dots, Y_{2d} \in Y$ be an orthonormal basis, then the element*

$$\mathcal{D} := \sum_{i=1}^{2d} Y_i \otimes Y_i \in \mathcal{A}$$

is called the (cubic) Dirac operator*.*

Lemma 4.3.4. *$\mathcal{D}$ is independent from the choice of basis.*

Proof. Let $X_1, \dots, X_{2d}$ be another orthonormal basis and let

$$Y_i = \sum_{i=1}^{2d} a_{ij} X_j,$$

then

$$\begin{aligned} \mathcal{D} &= \sum_{i=1}^{2d} Y_i \otimes Y_i \\ &= \sum_{i,j,k=1}^{2d} a_{ij} a_{ik} X_j \otimes X_k \\ &= \sum_{i,j=1}^{2d} X_i \otimes X_j \sum_{k=1}^{2d} a_{ki} a_{kj} \\ &= \sum_{i,j=1}^{2d} \delta_{ij} X_i \otimes X_j \\ &= \sum_{i=1}^{2d} X_i \otimes X_i. \end{aligned}$$

□

Theorem 4.3.5. *We have*

$$\mathcal{D}^2 = 1 \otimes \Omega_{\mathfrak{g}} - \Delta_Y(\Omega_{\mathfrak{r}}) + c$$

where $\Omega_{\mathfrak{h}}$ for $\mathfrak{h} \in \{\mathfrak{g}, \mathfrak{r}\}$ is the quadratic Casimir element of $\mathfrak{h}$ with respect to the invariant bilinear form B, and where $c \in \mathbb{C}$ is a constant.

Proof. See [Kos99, theorem 2.13]. □

Proposition 4.3.6. *The constant c from last theorem is equal to* $\frac{d^2}{8}$.

Proof. By [Kos99, theorems 1.81 and 2.13] and using that $v = 0$, we have $\alpha_Y(\Omega_\mathfrak{r}) = c$. By Section B.6, this equals $\frac{d^2}{8}$. □

In order to have $\mathcal{D}$ realistically be an operator, we'd like it to act on something, ideally some solutions to Ward identities. For that we should start by constructing $\mathcal{A}$-modules. It's not very surprising that $\mathcal{A}$-modules can be obtained by considering $S \otimes \mathcal{V}$ where S is a $\mathrm{C}\ell(Y)$-module (e.g. a spin module) and $\mathcal{V}$ is a $U(L)$-module. For example, we could consider $C^\infty(U; S \otimes V)$ where V is a vector space.

In order to talk about the Ward identities for such a space, we take V to be an MA-bimodule, and would now like to establish an MA-bimodule structure on $S \otimes V$ in such a way that this $MA \times MA$-action is compatible with the $\mathrm{C}\ell(Y)$-action in a suitable way.

Recall that α_Y satisfies

$$[\alpha_Y(\xi), v] = \mathrm{ad}(\xi)(v)$$

in $\mathrm{C}\ell(Y)$, for $\xi \in \mathfrak{r}, v \in Y$. If α_Y lifts to a group homomorphism $\phi_Y : MA \to \mathrm{Pin}(Y)$, this implies

$$\phi_Y(q) v \phi_Y(q^{-1}) = \mathrm{Ad}(q)(v)$$

for $q \in MA$, which then also holds for the continuation of $\mathrm{Ad}(q)$ onto all of $\mathrm{C}\ell(Y)$.

Lemma 4.3.7. *Let V be an MA-bimodule, S a $\mathrm{C}\ell(Y)$-module, then $S \otimes V$ can be turned into a $(\mathrm{C}\ell(Y), MA \times MA)$-module by*

$$\begin{aligned} \mathrm{C}\ell(Y) \ni p \cdot (s \otimes v) &:= (p \cdot s) \otimes v \\ MA \times MA \ni (p, q) \cdot (s \otimes v) &:= (\phi_Y(p)s) \otimes (p \cdot v \cdot q^{-1}). \end{aligned}$$

Note that the left copy of $MA \times MA$ acts on $\mathrm{C}\ell(Y)$ by

$$MA \xrightarrow{\mathrm{Ad}} O(Y) \to \mathrm{Aut}(\mathrm{C}\ell(Y)).$$

Proof. For compatibility let $p, p' \in MA$ and $q \in \mathrm{C}\ell(Y)$, as well as $s \in S, v \in V$. Then

$$\begin{aligned} (p,p') \cdot (q \cdot ((p,p')^{-1} \cdot (s \otimes v))) &= (p,p') \cdot (q \cdot (\phi_Y(p^{-1})s \otimes (p^{-1} \cdot v \cdot p'))) \\ &= (p,p') \cdot (q\phi_Y(p^{-1})s \otimes (p^{-1} \cdot v \cdot p')) \\ &= \phi_Y(p) q \phi_Y(p^{-1}) s \otimes v \\ &= \mathrm{Ad}(p)(q) s \otimes v \\ &= \mathrm{Ad}(p)(q) \cdot (s \otimes v), \end{aligned}$$

which is how (p, p') acts on q.

That shows the claim under the condition that α_Y lifts. To see that α_Y lifts, we note that $Y = \mathfrak{n} \oplus \overline{\mathfrak{n}}$ is a maximal isotropic decomposition, as $\mathfrak{n}, \overline{\mathfrak{n}}$ are both isotropic and dual to each other. This shows that $S := \bigwedge \overline{\mathfrak{n}}$ is the spin module, and if α_Y lifts when acting on S, it lifts to $\mathrm{Pin}(Y)$. Write $\pi : \mathrm{C}\ell(Y) \to \mathrm{End}(S)$ for the spin representation. First note that A is simply connected, so every smooth Lie algebra representation of A lifts to a Lie group representation, so we only need to check for M. We have

$$\begin{aligned}\pi(\alpha_Y(F_{\mu\nu})) &= \frac{1}{8}(\pi(P_\nu)\pi(K_\mu) - \pi(P_\mu)\pi(K_\nu)) \\ &= \frac{1}{4}(\epsilon_{P_\nu}\iota_{K_\mu} - \epsilon_{P_\mu}\iota_{K_\nu}).\end{aligned}$$

Note that

$$\begin{aligned}\frac{1}{4}\epsilon_{P_\nu}\iota_{K_\mu}P^{\rho_1}\wedge\cdots\wedge P^{\rho_r} &= \frac{1}{4}\epsilon_{P_\nu}\sum_{i=1}^{r}(-1)^i B(K_\mu, P^{\rho_i})P^{\rho_1}\wedge\cdots\widehat{P^{\rho_i}}\cdots\wedge P^{\rho_r} \\ &= \sum_{i=1}^{r}(-1)^i\delta_\mu^{\rho_i}P_\nu P^{\rho_1}\wedge\cdots\widehat{P^{\rho_i}}\cdots\wedge P^{\rho_r} \\ &= -\sum_{i=1}^{r}\delta_\mu^{\rho_i}P^{\rho_1}\wedge\cdots\widehat{P^{\rho_i}}P_\nu\cdots\wedge P^{\rho_r},\end{aligned}$$

so $\frac{1}{4}\epsilon_{P_\nu}\iota_{K_\mu}$ replaces every occurrence of P^μ with $-P_\nu$. This shows that $\pi(\alpha_Y(F_{\mu\nu}))$ replaces every occurrence of P^μ with $-P_\nu$, and every occurrence of P^ν with P_μ. Therefore,

$$\pi(\alpha_Y(F_{\mu\nu})) = \mathrm{ad}(F_{\mu\nu})$$

(extended to S). We therefore know for sure that the representation of $\mathfrak{m}$ on S lifts to a group representation of M, namely the extension of Ad. Since $\mathrm{C}\ell(Y) \cong \mathrm{End}(S)$, this shows that α_Y is the derivative of a Lie group homomorphism $\phi_Y : MA \to \mathrm{Pin}(Y)$. ☐

Lemma 4.3.8. *Let V be an MA-bimodule and S a $\mathrm{C}\ell(Y)$-module. Equip $S \otimes V$ with its $(\mathrm{C}\ell(Y), MA \times MA)$-module structure from Lemma 4.3.7. Then $C^\infty(U; S \otimes V)$ is an $(\mathcal{A}, MA \times MA)$-module.*

Proof. The compatibility of the actions of $U(L)$ and $MA \times MA$ was already the topic of Lemma 4.2.1, so it remains to show that the actions of $U(L)$ and $\mathrm{C}\ell(Y)$ commute, and that the actions of $\mathrm{C}\ell(Y)$ and $MA \times MA$ are compatible.

For the first let $\xi \in \mathfrak{g}, q \in \mathrm{C}\ell(Y), f \in C^\infty(U; S \otimes V)$, then

$$\begin{aligned}(\xi\cdot(q\cdot f))(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\,(q\cdot f)(\exp(-t\xi)x)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\,q\cdot f(\exp(-t\xi)x)\Big|_{t=0} \\ &= q\cdot\frac{\mathrm{d}}{\mathrm{d}t}\,f(\exp(-t\xi)x)\Big|_{t=0} \\ &= q\cdot(\xi\cdot f)(x) = (q\cdot(\xi\cdot f))(x).\end{aligned}$$

For the latter, let $q \in \mathrm{C}\ell(Y), p, p' \in MA, f \in C^\infty(U; S \otimes V)$. According to Lemma 4.3.7 we now have

$$\begin{aligned}((p,p')\cdot(q\cdot((p,p')^{-1}\cdot f)))(x) &= (p,p')\cdot(q\cdot((p,p')^{-1}\cdot f))(p^{-1}xp')\\ &= (p,p')\cdot(q\cdot((p,p')^{-1}\cdot f)(p^{-1}xp'))\\ &= (p,p')\cdot(q\cdot((p,p')^{-1}\cdot f(x)))\\ &= \mathrm{Ad}(p)(q)\cdot f(x)\\ &= (\mathrm{Ad}(p)(q)\cdot f)(x).\end{aligned}$$

□

Corollary 4.3.9. *Let V be an MA-bimodule and S a $\mathrm{C}\ell(Y)$-module, then* $\mathrm{Ward}(S \otimes V)$ *is a $\mathcal{A}^{MA}$-module.*

Proof. Follows from Lemma 4.3.8 and Lemma 2.1.3. □

Corollary 4.3.10. *In the same context, the Dirac operator $\mathcal{D}$ acts on* $\mathrm{Ward}(S \otimes V)$*.*

Proof. By Lemma 4.3.4, $\mathcal{D}$ is independent of the basis of Y, i.e. it is invariant under $O(Y)$. In particular, under $\mathrm{Ad}(MA)$, hence $\mathcal{D} \in \mathcal{A}^{MA}$. □

4.4. Which Dirac Operator?

Now, our treatment of the *Dirac operator* in the last section still leaves something to be desired. In a very algebraic setting we described an object whose square has as its leading term (filtering $\mathcal{A}$ by the filtration inherited purely from $U(\mathfrak{g})$) the quadratic Casimir element. However, when talking about the Dirac operator (e.g. in [Roe98]), one usually expects spin structures, Clifford bundles and the like. So does there exist something similar for our case?

Proposition 4.4.1. *Consider the manifold G/MA. Its tangent bundle is isomorphic to* $\mathrm{Ass}(Y)$*, the vector bundle associated to the principal MA-bundle $G \to MA$ and the MA-representation Y (adjoint). (Note that in this section all associated vector bundles, unless stated otherwise, will be assumed to refer to the group MA and the principal bundle $G \to MA\backslash G$.)*

Proof. Write $\pi : G \to MA\backslash G$ for the quotient map and define

$$\phi : \mathrm{Ass}(Y) \ni MA(g, X) \mapsto T_g\pi(T_eR_g((X)) \in T(MA\backslash G).$$

This map is well-defined as for $p \in MA$ we have $(g, X)p = (p^{-1}g, \mathrm{Ad}(p^{-1})(X))$ so that

$$\begin{aligned}T_{p^{-1}g}\pi(T_eR_{p^{-1}g}(\mathrm{Ad}(p^{-1})(X))) &= T_{p^{-1}g}\pi(T_eR_{p^{-1}g}(T_{p^{-1}}R_p(T_eL_{p^{-1}}(X))))\\ &= T_{p^{-1}g}\pi(T_{p^{-1}}R_g(T_eL_{p^{-1}}(X)))\\ &= T_{p^{-1}g}\pi(T_gL_{p^{-1}}(T_eR_g(X)))\\ &= T_g(\pi\circ L_{p^{-1}})(T_eR_g(X))\\ &= T_g\pi(T_eR_g(X)).\end{aligned}$$

Furthermore, since $T_g\pi$ and T_eR_g are linear maps, ϕ is a linear map on fibres, so it gives rise to a vector bundle morphism $\phi : \mathrm{Ass}(Y) \to T(MA\backslash G)$. Note that MA's action on G is free, so we have $\dim(MA\backslash G) = \dim(G) - \dim(MA) = \dim(\mathfrak{g}) - \dim(\mathfrak{r}) = \dim(Y)$. Thus it suffices to show fibrewise injectivity to show that ϕ is an isomorphism.

For that let $MAg \in MA\backslash G$ and let $X \in Y$ so that $\phi(MA(g, X)) = 0$. This means that the vector field $MAh \mapsto \phi((h, X)MA)$ is zero at $MAh = MAg$, so that its flow has to be constant when starting at MAg. As it turns out, this flow is $t \mapsto MA\exp(tX)g$, which is constant iff $\exp(tX) \in MA$ for all t. This in turn implies that $X \in \mathfrak{r} \cap Y$, which is trivial. □

Note that since B is $\mathfrak{r}$-invariant, it defines a metric (nondegenerate symmetric bilinear – not Hermitean) on the vector bundle $T(MA\backslash G)$.

Corollary 4.4.2. *The bundle* $\mathrm{C}\ell(T(MA\backslash G))$ *(whose fibre at* MAg *is* $\mathrm{C}\ell(T_{MAg}(MA\backslash G))$*) is isomorphic to* $\mathrm{Ass}(\mathrm{C}\ell(Y))$ *(with the representation again being the adjoint representation).*

Now take the representation $\phi_Y : MA \to \mathrm{Pin}(Y)$ from last section and make any Clifford module S into an MA-module using ϕ_Y. Then consider the vector bundle $\mathrm{Ass}(S)$.

Proposition 4.4.3. *Let* V *be an* MA*-bimodule, then the vector bundle* $\mathrm{Ass}(S \otimes V)$ *(consider only the left action on* V*) is a bundle of Clifford modules.*

Proof. The Clifford algebra $\mathrm{C}\ell(Y)$ acts on $S \otimes V$ by acting trivially on V, which lifts to a bundle morphism $\mathrm{C}\ell(T(MA\backslash G)) \otimes \mathrm{Ass}(S \otimes V) \to \mathrm{Ass}(S \otimes V)$ because

$$\mathrm{C}\ell(T(MA\backslash G)) \otimes \mathrm{Ass}(S \otimes V) \cong \mathrm{Ass}(\mathrm{C}\ell(Y)) \otimes \mathrm{Ass}(S \otimes V) \cong \mathrm{Ass}(\mathrm{C}\ell(Y) \otimes S \otimes V),$$

and for $q \in \mathrm{C}\ell(Y), s \in S, v \in V, p \in MA$ we have

$$\mathrm{Ad}(p)(q)\phi_Y(p)(s \otimes v) = \phi_Y(p)q\phi_Y(p)^{-1}\phi_Y(p)(s \otimes v) = \phi_Y(p)(q(s \otimes v)).$$ □

We now define a connection on the principal bundle $G \to MA\backslash G$, which we then use to define a connection on all these associated bundles. Define $\omega \in \Omega^1(G; \mathfrak{r})$

$$\omega_g(T_eR_g(\xi + \eta)) := \xi$$

for $\xi \in \mathfrak{r}, \eta \in Y$.

Proposition 4.4.4. *ω is a connection on the principal bundle* $G \to MA\backslash G$*.*

Proof. We need to check two properties: equivariance and that the infinitesimal action is mapped to itself.

For equivariance note that our "right" action of MA on G is left multiplication with the inverse, so instead of R_p, we are going to use $L_{p^{-1}}$:

$$\begin{aligned}(L_{p^{-1}}^*\omega)_g(T_eR_g(\xi+\eta)) &= \omega_{p^{-1}g}(T_gL_{p^{-1}}(T_eR_g(\xi+\eta)))\\ &= \omega_{p^{-1}g}(T_gL_{p^{-1}}(T_pR_{p^{-1}g}(T_eR_p(\xi+\eta))))\\ &= \omega_{p^{-1}g}(T_eR_{p^{-1}g}(T_pL_{p^{-1}}(T_eR_p(\xi+\eta))))\\ &= \omega_{p^{-1}g}(T_eR_{p^{-1}g}(\mathrm{Ad}(p^{-1})(\xi+\eta)))\\ &= \mathrm{Ad}(p^{-1})(\xi+\eta) = \mathrm{Ad}(p^{-1})(\omega_g(T_eR_g(\xi+\eta))).\end{aligned}$$

Furthermore, we have $\omega_g(T_eR_g(\xi)) = \xi$ for $\xi \in \mathfrak{r}$, where $g \mapsto T_eR_g(\xi)$ is the infinitesimal action of ξ. □

Let now V be any MA-module, then ω generates a connection $\mathrm{Ass}(V)$. If we interpret a section of $\mathrm{Ass}(V)$ as equivariant function on G, we have

$$\nabla s = \mathrm{d}s + \omega \cdot s$$

($\mathrm{d}s$ is taken component-wise, $\omega \cdot s$ is the action of $\mathfrak{r}$ on V). In particular, if we take the vector field X to be an equivariant function mapping to Y, we have

$$\nabla_X s = X(s)$$

(again, component-wise).

Proposition 4.4.5. *The induced connections on* $T(MA\backslash G) \cong \mathrm{Ass}(Y)$ *and* $\mathrm{Ass}(S \otimes V)$ *are compatible with the inner product induced from* B *and the Clifford action* $\mathrm{Ass}(\mathrm{C}\ell(Y)) \otimes \mathrm{Ass}(S \otimes V) \to \mathrm{Ass}(S \otimes V)$.

Proof. Let X, Y be vector fields on $MA\backslash G$, interpreted as equivariant functions, then

$$B(\nabla X, Y) + B(X, \nabla Y) = B(\mathrm{d}X\,, Y) + B(X, \mathrm{d}Y) + B(\omega \cdot X, Y) + B(X, \omega \cdot Y)$$

($\mathrm{d}X$ taken component-wise). Since B is $\mathfrak{r}$-invariant, multiplication with ω is skew-symmetric with respect to B, hence the above equals

$$B(\mathrm{d}X\,, Y) + B(Y, \mathrm{d}X) = \mathrm{d}B(X, Y).$$

Furthermore, by assumption $\omega(X) = \omega(Y) = 0$, hence we have

$$\nabla_X Y - \nabla_Y X = X(Y) - Y(X) = [X, Y],$$

so ∇ (on the tangent bundle) is the Levi-Civita connection.

Let now $X \in \Gamma(\mathrm{C}\ell(T(MA\backslash G))) \cong \Gamma(\mathrm{Ass}(\mathrm{C}\ell(Y)))$ and $s \in \Gamma(S \otimes V)$, interpreted as equivariant functions, then

$$\begin{aligned}\nabla(X \cdot s) &= \mathrm{d}(X \cdot s) + \omega \cdot (X \cdot s)\\ &= X \cdot \mathrm{d}s + \mathrm{d}X \cdot s + (\omega \cdot X) \cdot s + X \cdot (\omega \cdot s)\\ &= X \cdot (\mathrm{d}s + \omega \cdot s) + (\mathrm{d}X + \omega \cdot X) \cdot s\\ &= X \cdot \nabla s + \nabla X \cdot s.\end{aligned}$$

□

As a consequence, $\mathrm{Ass}(S \otimes V)$ is a Clifford bundle with compatible connection. By [Roe98, chapter 3] this is precisely the setting where we can define the Dirac operator (modulo the fact that B is not positive definite, so the analytical statements likely don't hold). Let $Y_1, \dots, Y_{2n}$ be a (complex) orthonormal basis of Y and let a local orthonormal frame of $T(MA \backslash G)$ be given by

$$X_i := \sum_{j=1}^{2n} a_{ij} Y_j \qquad (i = 1, \dots, 2n)$$

(where the a_{ij} are functions on an open MA-invariant subset of G). Then by definition, the Dirac operator is given as

$$s \mapsto \sum_{i=1}^{2n} X_i \cdot \nabla_{X_i}(s) = \sum_{i=1}^{2n} X_i \cdot X_i(s) = \sum_{i,j,k=1}^{2n} a_{ij} a_{ik} Y_j \cdot Y_k(s) = \sum_{i=1}^{2n} Y_i \cdot Y_i(s),$$

where $Y_i(s)$ refers to the coordinate-wise application of the right-invariant vector field whose value at the identity is Y_i, and $Y_i \cdot \dots$ refers to Clifford multiplication. Thus, we see that the Dirac operator on E corresponds to the action of

$$\sum_{i=1}^{2n} Y_i \otimes Y_i \in \mathrm{C}\ell(Y) \otimes U(L)$$

on functions $G \to S \otimes V$ that are equivariant on the left.

Next, we can define an action of MA on sections of $\mathrm{Ass}(S \otimes V)$. Let $p \in MA$ and $s \in \Gamma(\mathrm{Ass}(S \otimes V))$, then define

$$(s \cdot p)(MAg) := MA(v \cdot p, g) \qquad \text{where} \quad s(MAgp^{-1}) = MA(v, gp^{-1}),$$

in other words, if we interpret $s, s \cdot p$ as functions on G then $(s \cdot p)(g) = s(gp^{-1}) \cdot p$. In other words, the sections of $\mathrm{Ass}(S \otimes V)$ that are invariant under the MA-action are precisely the MA-biequivariant functions $G \to S \otimes V$. In particular, the sections $\Gamma(\mathrm{Ass}(S \otimes V), U/MA)$ that are MA-equivariant are exactly the solutions $\mathrm{Ward}(S \otimes V)$ to the conformal Ward identities, and the Dirac operator acting on invariant sections is precisely the same operator as the action described in Section 4.3.

5. Special Case: Scalar Conformal Blocks

As we already saw in Example 1.2.2, solutions to the conformal Ward identities become especially tractable if all (or as many as possible) M-actions involved are trivial. In this section we will now work out the differential operators that the quadratic Casimir element and the Dirac operator reduce to when applied to scalar solutions to the conformal Ward identities. More precisely: let $\Delta_1, \Delta_2 \in \mathbb{C}$ and let $V = \mathbb{C}$ be the MA-bimodule with

$$m \exp(\alpha D_0) \cdot v \cdot m' \exp(\beta D_0) = \exp(\alpha\Delta_1 + \beta\Delta_2)v.$$

Then we shall work with $\mathrm{Ward}(V)$ for the Casimir element, and with $\mathrm{Ward}(S \otimes V)$ for the Dirac operator (S a spin module).

5.1. Casimir Action

For the quadratic Casimir element, first note that a value of an equivariant function $f \in \mathrm{Ward}(V)$ is determined by just a few matrix entries:

Proposition 5.1.1. *Let $f \in \mathrm{Ward}(V)$, then*

$$f\begin{pmatrix} A & * & C \\ * & * & * \\ G & * & I \end{pmatrix} = \left|\frac{A-C+G-I}{2}\right|^{\frac{\Delta_1-\Delta_2}{2}} \left|\frac{A-C-G+I}{2}\right|^{-\frac{\Delta_1+\Delta_2}{2}} f_{\sigma\tau}(u,v),$$

where

$$u = \frac{4}{(A-I)^2-(C-G)^2} \qquad v = \frac{(C+G)^2-(A+I)^2}{(A-I)^2-(C-G)^2})$$

and

$$\sigma,\tau = \begin{cases} t,t & (A-C)^2 > (G-I)^2 \& 2u+2v+2uv > u^2+v^2+1 \\ t,s & (A-C)^2 > (G-I)^2 \& 2u+2v+2uv < u^2+v^2+1 \\ s,s & (A-C)^2 < (G-I)^2 \& 2u+2v+2uv > u^2+v^2+1 \\ s,t & (A-C)^2 < (G-I)^2 \& 2u+2v+2uv < u^2+v^2+1 \end{cases}.$$

Proof. By Section B.5, we see that the cross-ratios of the matrix in the claim are given by

$$u = \frac{4}{(A-I)^2-(C-G)^2}, \qquad v = \frac{(C+G)^2-(A+I)^2}{(A-I)^2-(C-G)^2}$$

and that the matrix can be brought into $Y_{\sigma\tau}$ by multiplication with $m\exp(\alpha D_0)$ on the left and $\exp(\beta D_0)m'$ on the right, with

$$\exp(\alpha-\beta)=\left|\frac{A-C+G-I}{2}\right|,\qquad \exp(-\alpha-\beta)=\left|\frac{A-C-G+I}{2}\right|,$$

which yields the expression in the claim. The only thing that remains is the correct choice for σ,τ. Recall from Proposition 3.3.9 that they reflect the signs of $\|x\|^2$, $\left\|y-\frac{\eta(x,y)}{\eta(x,x)}x\right\|^2$, respectively. We have

$$\frac{(A-C)^2-(G-I)^2}{4}=-x^2\exp(-2\beta),$$

which is positive iff $x^2<0$ and negative iff $x^2>0$. Similarly, we have

$$\begin{aligned}
\eta(\tilde{y},\tilde{y})&=\eta(y,y)-\frac{\eta(x,y)^2}{\eta(x,x)}\\
&=\frac{\eta(x,x)}{4}\left(4\frac{\eta(y,y)}{\eta(x,x)}-\frac{(\eta(x-y,x-y)-\eta(x,x)-\eta(y,y))^2}{\eta(x,x)^2}\right)\\
&=\frac{\eta(x,x)}{4}\left(4v-(u-v-1)^2\right)\\
&=\frac{\eta(x,x)}{4}(2u+2v+2uv-u^2-v^2-1),
\end{aligned}$$

so the sign of $\eta(\tilde{y},\tilde{y})$ depends on the sign of $\eta(x,x)$ and on the sign of $2u+2v+2uv-u^2-v^2-1$ in the way claimed. □

Proposition 5.1.2. *Let $f\in\mathrm{Ward}(V)$, then*

$$D_0\cdot f=-\Delta_1 f,\qquad F_{\mu\nu}\cdot f=0.$$

Proof. We have

$$\begin{aligned}
(D_0\cdot f)(x)&=\frac{\mathrm{d}}{\mathrm{d}t}\,f(\exp(-tD_0)x)\Big|_{t=0}\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\,t^{-\Delta_1}f(x)\Big|_{t=0}\\
&=-\Delta_1 f(x)\\
(F_{\mu\nu}\cdot f)(x)&=\frac{\mathrm{d}}{\mathrm{d}t}\,f(\exp(-tF_{\mu\nu})x)\Big|_{t=0}\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\,f(x)\Big|_{t=0}\\
&=0.
\end{aligned}$$

□

Using the results from Appendix B.6, we thus see that

$$\Omega_{\mathfrak{g}}\cdot f=\frac{\Delta_1(\Delta_1+d)}{2}f+\frac{1}{2}K^\mu P_\mu\cdot f.$$

To calculate the Casimir element, it therefore suffices to calculate the action of $K^\mu P_\mu$.

Proposition 5.1.3. *Up to order s^2, t^2, for $\mu = 1, \ldots, d$ (hence no summing)*

$$\exp(-tP_\mu)\exp(-sK^\mu)\xi_{\sigma\tau}(a,b) = \begin{pmatrix} A & * & C \\ * & * & * \\ G & * & I \end{pmatrix}$$

with

$$\begin{pmatrix} A & C \\ G & I \end{pmatrix} = w_0 \begin{pmatrix} 1+st & e_\mu^T(t\eta - s) & -st \\ -st & -e_\mu^T(t\eta + s) & 1+st \end{pmatrix} \begin{pmatrix} \frac{1\mp 1}{2} + \frac{1\mp v}{2u} & -\frac{1\mp 1}{2} + \frac{1\mp v}{2u} \\ \frac{u\mp v}{u} x^\bullet + w_0 w y^\bullet & -\frac{u\pm v}{u} x^\bullet + w_0 w y^\bullet \\ \frac{1\pm 1}{2} - \frac{1\pm v}{2u} & -\frac{1\pm 1}{2} - \frac{1\pm v}{2u} \end{pmatrix},$$

where x, y are such that $\xi_{\sigma\tau}(a,b) = \exp(x\cdot P) w \exp(y \cdot P)$.

Proof. We have

$$\exp(-tP_\mu)\exp(-sK^\mu) \begin{pmatrix} 1+st & e_\mu^T(t\eta - s) & -st \\ (s\eta - t)e_\mu & 1 + 2ste_\mu e_\mu^T & -(s\eta + t)e_\mu \\ -st & -e_\mu^T(t\eta + s) & 1+st \end{pmatrix}.$$

Similarly, we have

$$\xi_{\sigma\tau}(a,b) = \exp(x\cdot P) w \exp(y\cdot P) = \begin{pmatrix} w_0 \frac{1\mp 1 - y^2}{2} \pm \frac{w_0}{2}(y \mp w_0 w x)^2 & * & w_0 \frac{-(1\mp 1) - y^2}{2} \pm \frac{w_0}{2}(y \mp w_0 w x)^2 \\ w_0(1-y^2)x^\bullet + w y^\bullet & * & -w_0(1+y^2)x^\bullet + w y^\bullet \\ w_0 \frac{1\pm 1 - y^2}{2} \mp \frac{w_0}{2}(y \mp w_0 w x)^2 & * & w_0 \frac{-(1\pm 1) - y^2}{2} \mp \frac{w_0}{2}(y \mp w_0 w x)^2 \end{pmatrix}$$

(where $x^2 = \pm 1$). From (3.3) we know that

$$(y \mp w_0 w x)^2 = \frac{\pm 1}{u}, \qquad y^2 = \pm \frac{v}{u},$$

so that this becomes

$$\begin{pmatrix} w_0 \frac{1\mp 1}{2} \mp \frac{w_0 v}{2u} + \frac{w_0}{2u} & * & -w_0 \frac{1\mp 1}{2} \mp \frac{w_0 v}{2u} + \frac{w_0}{2u} \\ w_0 \frac{u \mp v}{u} x^\bullet + w y^\bullet & * & -w_0 \frac{u\pm v}{u} x^\bullet + w y^\bullet \\ w_0 \frac{1\pm 1}{2} \mp \frac{w_0 v}{2u} - \frac{w_0}{2u} & * & -w_0 \frac{1\pm 1}{2} \mp \frac{w_0 v}{2u} - \frac{w_0}{2u} \end{pmatrix}.$$

Consequently, the matrix $\exp(-tP_\mu)\exp(-sK^\mu)\xi_{\sigma\tau}(a,b)$ has in its corners the entries

$$\begin{pmatrix} A & C \\ G & I \end{pmatrix} = w_0 \begin{pmatrix} 1+st & e_\mu^T(t\eta - s) & -st \\ -st & -e_\mu^T(t\eta + s) & 1+st \end{pmatrix} \begin{pmatrix} \frac{1\mp 1}{2} + \frac{1\mp v}{2u} & -\frac{1\mp 1}{2} + \frac{1\mp v}{2u} \\ \frac{u\mp v}{u} x^\bullet + w_0 w y^\bullet & -\frac{u\pm v}{u} x^\bullet + w_0 w y^\bullet \\ \frac{1\pm 1}{2} - \frac{1\pm v}{2u} & -\frac{1\pm 1}{2} - \frac{1\pm v}{2u} \end{pmatrix}.$$

We now distinguish cases. We have either $x\cdot P = P_1$ or P_μ, hence $x^\bullet = e_1$ or e_d. Similarly, $y \cdot P$ is a linear combination of P_1, P_2, P_{d-1}, P_d in terms of a, b, so then $y = ae_1 + be_2$, or any of the other combinations (without minus signs).

In the case $\sigma\tau = ss$, this reads

$$w_0\begin{pmatrix}1+st & -(s+t) & -st\\ -st & t-s & 1+st\end{pmatrix}\begin{pmatrix}\frac{1-v}{2u} & \frac{1-v}{2u}\\ \frac{u-v}{u}e_1 + w_0(w_1ae_1+w_2be_2) & -\frac{u+v}{u}e_1 + w_0(w_1ae_1+w_2be_2)\\ \frac{2u-1-v}{2u} & \frac{-2u-1-v}{2u}\end{pmatrix}$$

$$=w_0\begin{pmatrix}1+st & -(s+t) & -st\\ -st & t-s & 1+st\end{pmatrix}\begin{pmatrix}\frac{1}{2}-w_0w_1a & \frac{1}{2}-w_0w_1a\\ A\delta_{\mu,1}+w_0w_2b\delta_{\mu,2} & (A-2)\delta_{\mu,1}+w_0w_2b\delta_{\mu,2}\\ \frac{1}{2}+w_0w_1a-a^2-b^2 & -\frac{3}{2}+w_0w_1a-a^2-b^2\end{pmatrix}$$

where $A = w_0w_1a + 1 - a^2 - b^2$.

For $\sigma\tau = st$, it reads

$$w_0\begin{pmatrix}1+st & e_\mu^T(t\eta-s) & -st\\ -st & -e_\mu^T(t\eta+s) & 1+st\end{pmatrix}\begin{pmatrix}\frac{1-v}{2u} & \frac{1-v}{2u}\\ \frac{u-v}{u}e_1 + w_0(w_1ae_1+w_dbe_d) & -\frac{u+v}{u}e_1 + w_0(w_1ae_1+w_dbe_d)\\ \frac{2u-1-v}{2u} & \frac{-2u-1-v}{2u}\end{pmatrix}$$

$$=w_0\begin{pmatrix}1+st & e_\mu^T(t\eta-s) & -st\\ -st & -e_\mu^T(t\eta+s) & 1+st\end{pmatrix}\begin{pmatrix}\frac{1}{2}-w_0w_1a & \frac{1}{2}-w_0w_1a\\ Ae_1+w_0w_dbe_d & (A-2)e_1+w_0w_dbe_d\\ \frac{1}{2}+w_0w_1a-a^2+b^2 & -\frac{3}{2}+w_0w_1a-a^2+b^2\end{pmatrix}$$

where $A = w_0w_1a + 1 - a^2 + b^2$.

For $\sigma\tau = ts$, it reads

$$w_0\begin{pmatrix}1+st & e_\mu^T(t\eta-s) & -st\\ -st & -e_\mu^T(t\eta+s) & 1+st\end{pmatrix}\begin{pmatrix}\frac{2u+1+v}{2u} & \frac{-2u+1+v}{2u}\\ -\frac{u+v}{u}e_d + w_0(w_dae_d+w_1be_1) & \frac{u-v}{u}e_d + w_0(w_dae_d+w_1be_1)\\ -\frac{1-v}{2u} & -\frac{1-v}{2u}\end{pmatrix}$$

$$=w_0\begin{pmatrix}1+st & e_\mu^T(t\eta-s) & -st\\ -st & -e_\mu^T(t\eta+s) & 1+st\end{pmatrix}\begin{pmatrix}\frac{3}{2}+w_0w_da+a^2-b^2 & -\frac{1}{2}+w_0w_da+a^2-b^2\\ Ae_d+w_0w_1be_1 & (A-2)e_d+w_0w_1be_1\\ -\frac{1}{2}-w_0w_da & -\frac{1}{2}-w_0w_da\end{pmatrix}$$

where $A = w_0w_da + 1 + a^2 - b^2$.

And for $\sigma\tau = tt$ we get

$$w_0\begin{pmatrix}1+st & -(s+t) & -st\\ -st & t-s & 1+st\end{pmatrix}\begin{pmatrix}\frac{2u+1+v}{2u} & \frac{-2u+1+v}{2u}\\ A\delta_{\mu,d}+w_0w_{d-1}b\delta_{\mu,d-1} & (A-2)\delta_{\mu,d}+w_0w_{d-1}b\delta_{\mu,d-1}\\ -\frac{1-v}{u} & -\frac{1-v}{2u}\end{pmatrix}$$

$$=w_0\begin{pmatrix}1+st & -(s+t) & -st\\ -st & t-s & 1+st\end{pmatrix}\begin{pmatrix}\frac{3}{2}+w_0w_da+a^2+b^2 & -\frac{1}{2}+w_0w_da+a^2+b^2\\ A\delta_{\mu,d}+w_0w_{d-1}b\delta_{\mu,d-1} & (A-2)\delta_{\mu,d}+w_0w_{d-1}b\delta_{\mu,d-1}\\ -\frac{1}{2}-w_0w_da & -\frac{1}{2}-w_0w_da\end{pmatrix}$$

for $A = w_0w_da + 1 + a^2 + b^2$. □

Corollary 5.1.4. *For $f \in \mathrm{Ward}(V)$ we have*

$$(\Omega_{\mathfrak{g}}\cdot f)_{\sigma\tau}(u,v) = \Bigg((1-u-v)\partial_v\Big(v\partial_v - \frac{\Delta_1+\Delta_2}{2}\Big) + 2u\partial_u\Big(u\partial_u + \Delta_2 - \frac{d}{2}\Big)$$

$$(v-u-1)(u\partial_u+v\partial_v)\Big(u\partial_u+v\partial_v - \frac{\Delta_1-\Delta_2}{2}\Big) + \frac{\Delta_2^2 - d\Delta_2}{2}\Bigg)f_{\sigma\tau}(u,v)$$

Proof. Plugging the expressions from Proposition 5.1.3 into the expression from Proposition 5.1.1 and differentiating with respect to s, t at $s = t = 0$ gives

$$(K^\mu P_\mu \cdot f)_{\sigma\tau}(u, v).$$

Curiously, for all choices of σ, τ we get the same expressions but potentially for different choices of μ. To unify them, write $\xi_{\sigma\tau}(a, b) = \exp(x \cdot P) w \exp(y \cdot P)$. Then exactly one component of x is one, say $x^\mu = 1$, and exactly one other component of y is nonzero, say $y^\nu = b$.

For $\rho \notin \{\mu, \nu\}$ we get

$$\begin{aligned}
(K^\mu P_\mu \cdot f)_{\sigma\tau}(u, v) &= A f_{\sigma\tau}(u, v) - (\Delta_1 + \Delta_2) f_{\sigma\tau}(u, v) - 2u\partial_u f_{\sigma\tau}(u, v) \\
(K^\nu P_\nu \cdot f)_{\sigma\tau}(u, v) &= B f_{\sigma\tau}(u, v) - (\Delta_1 + \Delta_2) f_{\sigma\tau}(u, v) - 2u\partial_u f_{\sigma\tau}(u, v) \\
(K^\rho P_\rho \cdot f)_{\sigma\tau}(u, v) &= -(\Delta_1 + \Delta_2) f_{\sigma\tau}(u, v) - 2u\partial_u f_{\sigma\tau}(u, v)
\end{aligned}$$

(no summation over repeated indices), for differential operators

$$\begin{aligned}
A = {} & 2(v - 1 - u)\Big((u\partial_u)^2 + 2u\partial_u v\partial_v - \frac{\Delta_1 - \Delta_2}{2}(u\partial_u + v\partial_v)\Big) + 4u\partial_u(u\partial_u + \Delta_2) \\
& + (v - 1 - u)(v + u - 1)\partial_v v\partial_v - (1 + u^2 + v^2 - 2u - 2v - 2uv)u\partial_u\partial_v \\
& - (\Delta_1 + \Delta_2)(1 - u - v)\partial_v + \Delta_2^2 - \Delta_1^2 \\
B = {} & (1 + u^2 + v^2 - 2u - 2v - 2uv)(\partial_v v\partial_v + u\partial_u\partial_v).
\end{aligned}$$

Summing them all together yields the claimed expression. □

We will see later how exactly to make sense of this differential operator.

5.2. Dirac Action

In order to get an action by the Dirac operator, we now have to break the initial premise of this section and make our functions vector-valued. Instead of $\mathrm{Ward}(V)$ we're now considering $\mathrm{Ward}(S \otimes V)$. Since we're now working with vector-valued functions, we can't use the trick from last section where the value of the function only depends on four matrix entries. However, since we're only working with first-order differential operators, it is feasible to work with the $\overline{N}NMA$ decomposition and its cocycles. This has the side-effect of making our calculations valid for other MA-bimodules as well.

From Appendix B.6 we know that $K^1, \ldots, K^d, P^1, \ldots, P^d$ and

$$\frac{1}{4}P_1, \ldots, \frac{1}{4}P_d, \frac{1}{4}K_1, \ldots, \frac{1}{4}K_d$$

are dual bases of Y, hence the cubic Dirac operator is given

$$\mathcal{D} = \frac{1}{4}K^\mu \otimes P_\mu + \frac{1}{4}P^\mu \otimes K_\mu.$$

Both summands happen to be MA-invariant, so that we shall calculate them separately: $4\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$.

5.2.1. $\mathcal{D}_1$

We're now calculating a $C\ell(Y)$-valued differential operator, where the differential part comes from the action of P_ρ.

For any vector $x \in \mathbb{R}^{p,q}$ write $\|x\| := \sqrt{|\eta(x,x)|}$. Throughout this section let $\sigma, \tau \in \{s,t\}$, let $x, y \in \mathbb{R}^{p,q}$ be such that

$$\xi_{\sigma\tau}(a,b) = \exp(x \cdot P) w \exp(y \cdot P),$$

in particular $x = e_\mu, y = ae_\mu + be_\nu$ for appropriate indices μ, ν. In this section we will assume no summation over repeated occurrences of μ, ν. Moreover, all equations involving t are understood to hold up to $\mathcal{O}(t^2)$.

Proposition 5.2.1. *Let $\rho = 1, \dots, d$, write $\alpha(t) := \log(\|x - te_\mu\|)$ and let $m(t) \in M$ be such that*

$$m(t)x = \frac{x - te_\rho}{\|x - te_\rho\|}$$

(equivalently $\exp(\alpha(t))m(t)x = x - te_\rho$) and

$$\exp(\alpha(t))c_w(m(t))^{-1}y = y'(t) = a'(t)e_\mu + b'(t)e_\nu.$$

all up to order $\mathcal{O}(t^2)$. Then

$$\exp(-tP_\rho)\xi_{\sigma\tau}(a,b) = \exp(-\alpha(t)D_0)m(t)\xi_{\sigma\tau}(a'(t),b'(t))c_w(m(t))^{-1}\exp(-\alpha(t)D_0).$$

In particular, if $u'(t), v'(t)$ are the cross-ratios associated to $\xi_{\sigma\tau}(a'(t),b'(t))$, we get

$$(\exp(tP_\mu)\cdot f)_{\sigma\tau}(u,v) = \exp(-\alpha(t)D_0)m(t)\cdot f_{\sigma\tau}(u'(t),v'(t))\cdot c_w(m(t))^{-1}\exp(-\alpha(t)D_0)$$

for $f \in \mathrm{Ward}(S \otimes V)$ (for any MA-bimodule V).

Proof. We identify a vector v with the column vector $v^\bullet$, so that $v = v^\alpha e_\alpha$. We thus obtain

$$\begin{aligned}\exp(-tP_\rho)\xi_{\sigma\tau}(a,b) &= \exp(-tP_\rho)\exp(x\cdot P)w\exp(y\cdot P)\\ &= \exp((x - te_\rho)\cdot P)w\exp(y\cdot P).\end{aligned}$$

Now, we have $(x - te_\rho)\cdot P = \mathrm{Ad}(\exp(-\alpha(t)D_0)m(t))(x\cdot P)$ by assumption, hence

$$\begin{aligned}\exp(-tP_\rho)\xi_{\sigma\tau}(a,b) &= \exp(-\alpha(t)D_0)m(t)\exp(x\cdot P)\exp(\alpha(t)D_0)m(t)^{-1}w\exp(y\cdot P)\\ &= \exp(-\alpha(t)D_0)m(t)\exp(x\cdot P)w\exp(-\alpha(t)D_0)c_w(m(t))^{-1}\exp(y\cdot P)\\ &= \exp(-\alpha(t)D_0)m(t)\exp(x\cdot P)w\exp\Big((\alpha(t)c_w(m(t))^{-1}y)\cdot P\Big)\cdot\\ &\quad\exp(-\alpha(t)D_0)c_w(m(t))^{-1}\\ &= \exp(-\alpha(t)D_0)m(t)\exp(x\cdot P)w\exp(y'(t)\cdot P)\exp(-\alpha(t)D_0)c_w(m(t))^{-1}\\ &= \exp(-\alpha(t)D_0)m(t)\xi_{\sigma\tau}(a'(t),b'(t))\exp(-\alpha(t)D_0)c_w(m(t))^{-1}.\end{aligned}$$

□

Proposition 5.2.2 (Case $\rho = \mu$)**.** *For $f \in \mathrm{Ward}(S \otimes V)$ (V an arbitrary MA-bimodule) we have*

$$(1 \otimes P_\mu \cdot f)_{\sigma\tau}(u,v) = D_0 \cdot f_{\sigma\tau}(u,v) + f_{\sigma\tau}(u,v) \cdot D_0 + (v+1-u)u\partial_u f_{\sigma\tau}(u,v) + (v-1-u)v\partial_v f_{\sigma\tau}(u,v).$$

Proof. We have $\|x - te_\mu\| = 1 - t$, so that $\alpha(t) = -t$. Pick $m(t) = 1$, then we can apply Proposition 5.2.1 to get

$$(\exp(tP_\mu) \cdot f)_{\sigma\tau}(u,v) = \exp(tD_0) \cdot f_{\sigma\tau}(u'(t), v'(t)) \cdot \exp(tD_0)$$

where $a'(t) = (1-t)a, b'(t) = (1-t)b)$, so that

$$\begin{aligned}
u'(t) &= \begin{cases} \frac{1}{(a-at-w_0w_1)^2 \pm (1-t)^2 b^2} & \sigma = s \\ \frac{1}{(a-at+w_0w_d)^2 \mp (1-t)^2 b^2} & \sigma = t \end{cases} \\
&= \frac{1}{u^{-1} - t(u^{-1}v + u^{-1} - 1)} = \frac{u}{1 - t(v+1-u)} \approx u + tu(v+1-u) \\
\frac{v'(t)}{u'(t)} &= (1-t)^2(a^2 \pm b^2) = (1-t)^2 \frac{v}{u} \\
v'(t) &= \frac{(1-2t)v}{1-t(v+1-u)} \approx v + tv(v-1-u).
\end{aligned}$$

Hence,

$$(P_\mu \cdot f)_{\sigma\tau}(u,v) = D_0 \cdot f_{\sigma\tau}(u,v) + f_{\sigma\tau}(u,v) \cdot D_0 + ((v+1-u)u\partial_u + (v-1-u)v\partial_v)f_{\sigma\tau}(u,v).$$

□

Proposition 5.2.3 (Case $\rho = \nu$)**.** *For $f \in \mathrm{Ward}(S \otimes V)$ (V arbitrary MA-bimodule) we have*

$$(1 \otimes P_\nu \cdot f)_{\sigma\tau} = -F_\nu{}^\mu \cdot f_{\sigma\tau}(u,v) + w_\mu w_\nu f_{\sigma\tau} \cdot F_\nu{}^\mu - 2\eta_{\nu\nu} w_0 w_\nu b(u\partial_u + v\partial_v) f_{\sigma\tau}(u,v).$$

Proof. We have $\alpha(t) = 0$. Next note that $F_\nu{}^\mu e_\mu = F_\nu{}^\mu x = e_\nu$ (no summation over μ), so that $m(t) = 1 - tF_\nu{}^\mu$ satisfies $m(t)x = x - te_\nu$, so we can use it. Furthermore,

$$\begin{aligned}
\exp(\alpha(t))c_w(m(t))^{-1}y &= (1 + w_\mu w_\nu t F_\nu{}^\mu)(ae_\mu + be_\nu) \\
&= (b + w_\mu w_\nu at)e_\nu + (a - w_\mu w_\nu \eta_{\mu\mu}\eta_{\nu\nu} bt)e_\mu,
\end{aligned}$$

so that $a'(t) = a - w_\mu w_\nu \eta_{\mu\mu}\eta_{\nu\nu} bt$ and $b'(t) = b + w_\mu w_\nu at$. Then

$$u'(t) = u(1 - 2\eta_{\nu\nu} w_0 w_\mu ubt), \qquad v'(t) = v(1 - 2\eta_{\nu\nu} w_0 w_\mu ubt),$$

which shows the claim. □

Proposition 5.2.4 (Other Cases)**.** *Let $\rho \notin \{\mu\nu\}$ and let $f \in \mathrm{Ward}(S \otimes V)$ for an arbitrary MA-bimodule V, then*

$$(1 \otimes P_\rho \cdot f)_{\sigma\tau}(u,v) = -\left(F_\rho{}^\mu - w_\mu w_\nu \frac{a}{b} F_\rho{}^\nu\right) \cdot f_{\sigma\tau}(u,v) + w_\rho w_\mu f_{\sigma\tau}(u,v) \cdot \left(F_\rho{}^\mu - \frac{a}{b} F_\rho{}^\nu\right)$$

Proof. We have $\alpha(t) = 0$. Then $m(t) = 1 - tF_\rho{}^\mu + w_\mu w_\nu \frac{a}{b} F_\rho{}^\nu$ maps

$$m(t)x = 1 - te_\rho.$$

Furthermore, we have

$$c_w(m(t))^{-1} = w\Big(1 + tF_\rho{}^\mu - w_\mu w_\nu \frac{a}{b} F_\rho{}^\nu\Big)w = 1 + tw_\rho w_\mu \Big(F_\rho{}^\mu - \frac{a}{b} F_\rho{}^\nu\Big),$$

which maps $ae_\mu + b_\nu$ to itself. Thus, we obtain $u'(t) = u, v'(t) = v$ and hence

$$(1 \otimes P_\rho \cdot f)_{\sigma,\tau}(u,v) = -\Big(F_\rho{}^\mu - w_\mu w_\nu \frac{a}{b} F_\rho{}^\nu\Big) \cdot f_{\sigma\tau}(u,v) + w_\rho w_\mu f_{\sigma\tau}(u,v) \cdot \Big(F_\rho{}^\mu - \frac{a}{b} F_\rho{}^\nu\Big),$$

as claimed. □

In the case where $V = \mathbb{C}$ with D_0 acts by Δ_1, Δ_2 from the left and right, respectively, we obtain the following differential operators

$$\begin{aligned}
(1 \otimes P_\mu \cdot f)_{\sigma\tau}(u,v) &= (\Delta_1 + \Delta_2 + \alpha_Y(D_0) + (v + 1 - u)u\partial_u + (v - 1 - u)v\partial_v) f_{\sigma\tau}(u,v) \\
(1 \otimes P_\nu \cdot f)_{\sigma\tau}(u,v) &= (-\alpha_Y(F_\nu{}^\mu) - 2\eta_{\nu\nu} w_0 w_\nu ub(u\partial_u + v\partial_v)) f_{\sigma\tau}(u,v) \\
(1 \otimes P_\rho \cdot f)_{\sigma\tau}(u,v) &= -\Big(\alpha_Y(F_\rho{}^\mu) - w_\mu w_\nu \frac{a}{b} \alpha_Y(F_\rho{}^\nu)\Big) f_{\sigma\tau}(u,v).
\end{aligned}$$

Inserting the well-known expressions for $\alpha_Y(D_0)$ and $\alpha_Y(F_{\mu\nu})$, we obtain

$$\begin{aligned}
(\mathcal{D}_1 \cdot f)_{\sigma\tau}(u,v) =& K^\mu \Big(\Delta_1 + \Delta_2 - \frac{d}{2} + (v + 1 - u)u\partial_u + (v - 1 - u)v\partial_v\Big) f_{\sigma\tau}(u,v) \\
& -2ubK_\nu w_\nu w_0 (u\partial_u + v\partial_v) f_{\sigma\tau}(u,v) \\
& + w_\mu w_\nu \frac{a}{b} K^\nu \Big(\alpha_Y(D_0) + \frac{d}{2} + \frac{K^\mu P_\mu}{8}\Big) f_{\sigma\tau}(u,v)
\end{aligned}$$

5.2.2. $\mathcal{D}_2$

All assumptions from the last section about x, y, μ, ν, the Einstein summation convention, and equations involving t are also in force in this section.

In this section we're going to use heavily that

$$\exp(bt \cdot K) \exp\big(x\dot{P}\big) = \exp(x' \cdot P) \exp(b' \cdot P) m \exp(\alpha D_0)$$

where

$$\begin{aligned}
\alpha &= 2tb_\mu x^\mu \\
x'^\mu &= (1 - 2tb_\nu x^\nu)x^\mu + x^2 tb^\mu \\
b'^\mu &= tb^\mu \\
m^\mu{}_\nu &= \delta^\mu_\nu + 2tb^\mu x_\nu - 2tx^\mu b_\nu
\end{aligned}$$

(see Proposition B.4.1). In particular, for $x = e_\mu$ and $b = -e^\rho$ we obtain

$$\begin{aligned}
\alpha &= -2t\delta^\rho_\mu \\
x' &= e_\mu + \eta_{\mu\mu}t(2\delta^\rho_\mu - 1)e^\rho \\
b' &= -te^\rho \\
m &= 1 - 2tF^\rho{}_\mu .
\end{aligned}$$

Proposition 5.2.5. *Let $\rho = 1, \ldots, d$. Write $\alpha(t) := \log(\|x + te_\rho\|)$ and $m'(t) = 1 - 2tF^\rho{}_\mu$. Assume, $m(t) \in M$ is such that*

$$m(t)x = \frac{x + \eta_{\mu\mu}t(2\delta^\rho_\mu - 1)e^\rho}{\|\cdots\|},$$

(equivalently $\exp(\alpha(t))m(t)x = x + \eta_{\mu\mu}t(2\delta^\rho_\mu - 1)e^\rho$) and such that

$$\exp(-\alpha(t))c_w(m^{-1}m')(y + w_0 w_\rho te^\rho) = y'(t) = a'(t)e_\mu + b'(t)e_\nu$$

lies in the span of e_μ, e_ν. Then

$$\exp(-tK^\rho)\xi_{\sigma\tau}(a,b) = \exp(-\alpha(t)D_0)m(t)\xi_{\sigma\tau}(a',b')c_w(m^{-1}m')\exp(\alpha(t)D_0),$$

hence for any $f \in \mathrm{Ward}(S \otimes V)$ (arbitary V) we have

$$(\exp(tK^\rho)\cdot f)_{\sigma\tau}(u,v) = \exp(-\alpha(t)D_0)m(t)\cdot f_{\rho\sigma}(u',v')\cdot c_w(m^{-1}m')\exp(\alpha(t)D_0).$$

Proof. We have

$$\exp(-tK^\rho)\exp(P_\mu) = \exp\Big(P_\mu + \eta_{\mu\mu}t(2\delta^\rho_\mu - 1)P^\rho\Big)\exp(-tK^\rho)m'(t)\exp(-2\alpha(t)D_0).$$

Note that the vector

$$e_\mu + \eta_{\mu\mu}t(2\delta^\rho_\mu - 1)e^\rho$$

has "norm" $1 + t$ if $\rho = \mu$ and 1 otherwise, so we have

$$\exp(\alpha(t))m(t)x = x + \eta_{\mu\mu}t(2\delta^\rho_\mu - 1)e^\rho$$

or in other words,

$$\exp(-tK^\rho)\exp(P_\mu) = \exp(-\alpha(t)D_0)m\exp(P_\mu)m(t)^{-1}m'(t)\exp(-\alpha(t)D_0)\exp(-tK^\rho)$$

(where we use that $-tK^\rho$ is already purely of order t, so it commutes with $m'(t)$ and $\exp(-\alpha(t)D_0)$). Thus,

$$\begin{aligned}
\exp(-tK^\rho)\xi_{\sigma\tau}(a,b) &= \exp(-\alpha(t)D_0)m\exp(P_\mu)wc_w(m^{-1}m')\exp(\alpha(t)D_0)\exp((y + w_0 w_\rho te^\rho)\cdot P) \\
&= \exp(-\alpha(t)D_0)m\exp(P_\mu)w\exp(y'(t)\cdot P)c_w(m^{-1}m')\exp(\alpha(t)D_0) \\
&= \exp(-\alpha(t)D_0)m\xi_{\sigma\tau}(a'(t),b'(t))c_w(m^{-1}m')\exp(\alpha(t)D_0). \qquad \square
\end{aligned}$$

Proposition 5.2.6 (Case $\rho = \mu$)**.** *For $f \in \mathrm{Ward}(S \otimes V)$ (V arbitrary MA-bimodule) we have*

$$(1 \otimes K^\mu \cdot f)_{\sigma\tau}(u,v) = -D_0 \cdot f_{\sigma\tau}(u,v) + f_{\sigma\tau}(u,v) \cdot D_0 + 2u\partial_u f_{\sigma\tau}(u,v) \\ - (1-u-v)\partial_v f_{\sigma\tau}(u,v).$$

Proof. We have $\alpha(t) = t$ and we can pick $m' = m = 1$. Then we obtain

$$\exp(-tK^\mu)\xi_{\sigma\tau}(a,b) = \exp(-tD_0)\xi_{\sigma\tau}((1-t)(a + w_0 w_\mu \eta_{\mu\mu} t), (1-t)b). \exp(tD_0),$$

The parameters $a' = (1-t)a + w_0 w_\mu \eta_{\mu\mu} t$ and $b' = (1-t)b$ correspond to the cross-ratios

$$u = (1+2t)u, \qquad v = v + t(v-1-u),$$

so that we obtain

$$(1 \otimes K^\mu \cdot f)_{\sigma\tau}(u,v) = -D_0 \cdot f_{\sigma\tau}(u,v) + f_{\sigma\tau}(u,v) \cdot D_0 + (2u\partial_u + (v-1-u)\partial_v) f_{\sigma\tau}(u,v). \ \square$$

Proposition 5.2.7 (Case $\rho = \nu$)**.** *For $f \in \mathrm{Ward}(S \otimes V)$ (V arbitrary MA-bimodule) we have*

$$(1 \otimes K^\nu \cdot f)_{\sigma\tau}(u,v) = {F_\mu}^\nu \cdot f_{\sigma\tau}(u,v) + w_\mu w_\nu f_{\sigma\tau}(u,v) \cdot {F_\mu}^\nu + 2b\eta_{\mu\mu} w_0 w_\nu u \partial_v f_{\sigma\tau}(u,v).$$

Proof. We have $\alpha(t) = 0$ and $m'(t) = 1 - 2t{F^\nu}_\mu$. Pick $m(t) = 1 - t{F^\nu}_\mu$. This maps $e_\mu = x$ to $e_\mu - t\eta_{\mu\mu} e^\nu$, which is what we want. Furthermore, we have

$$c_w(m^{-1}m') = c_w(1 - t{F^\nu}_\mu) = 1 - w_\mu w_\nu t {F^\nu}_\mu,$$

which maps $y + w_0 w_\nu t e^\nu$ to

$$(a + w_\mu w_\nu bt)e_\mu + (b + w_\nu \eta_{\nu\nu} t(w_0 - w_\mu \eta_{\mu\mu} a))e_\nu,$$

so that

$$a' = a + w_\mu w_\nu bt, \qquad b' = b + w_\nu \eta_{\nu\nu} t(w_0 - w_\mu \eta_{\mu\mu} a).$$

This shows that the new cross-ratios are $u'(t) = u$ and $v'(t) = v + 2ub\eta_{\mu\mu} w_0 w_\nu t$. Thus,

$$(1 \otimes K^\nu \cdot f)_{\sigma\tau}(u,v) = {F_\mu}^\nu \cdot f_{\sigma\tau}(u,v) + w_\mu w_\nu f_{\sigma\tau}(u,v) \cdot {F_\mu}^\nu \\ + 2ub\eta_{\mu\mu} w_0 w_\nu \partial_v f_{\sigma\tau}(u,v). \qquad \square$$

Proposition 5.2.8 (Other Cases)**.** *For $f \in \mathrm{Ward}(S \otimes V)$ (V arbitrary MA-bimodule) and $\rho \notin \{\mu, \nu\}$ we have*

$$(1 \otimes K^\rho \cdot f)_{\sigma\tau}(u,v) = -\left({F^\rho}_\mu + w_\mu w_\nu c F^{\rho\nu}\right) \cdot f_{\sigma\tau}(u,v) \\ - w_\rho w_\mu f_{\sigma\tau}(u,v) \cdot \left({F^\rho}_\mu - cF^{\rho\nu}\right).$$

where $c = \frac{\eta_{\mu\mu} a - w_0 w_\mu}{b}$

Proof. We have $\alpha(t) = 0$ and $m'(t) = 1 - 2tF^{\rho}{}_{\mu}$. Pick

$$m(t) = 1 - tF^{\rho}{}_{\mu} - w_\mu w_\nu \frac{\eta_{\mu\mu}a - w_0 w_\mu}{b} tF^{\rho\nu}.$$

Then $m(t)x = x - \eta_{\mu\mu} t e^{\rho}$ and

$$\begin{aligned} c_w(m(t)^{-1}m') &= w\left(1 - tF^{\rho}{}_{\mu} + w_\mu w_\nu \frac{\eta_{\mu\mu}a - w_0 w_\mu}{b} tF^{\rho\nu}\right) w \\ &= 1 - w_\rho w_\mu t\left(F^{\rho}{}_{\mu} - \frac{\eta_{\mu\mu}a - w_0 w_\mu}{b} tF^{\rho\nu}\right) \end{aligned}$$

which maps $ae_\mu + be_\nu + w_0 w_\rho t e^{\rho}$ to $ae_\mu + be_\nu$, hence the cross-ratios are unaltered. Thus we obtain

$$\begin{aligned} (1 \otimes K^{\rho} \cdot f)_{\sigma\tau}(u,v) = &- \left(F^{\rho}{}_{\mu} + w_\mu w_\nu c F^{\rho\nu}\right) \cdot f_{\sigma\tau}(u,v) \\ &- w_\rho w_\mu f_{\sigma\tau}(u,v) \cdot \left(F^{\rho}{}_{\mu} - cF^{\rho\nu}\right) \end{aligned}$$

where

$$c = \frac{\eta_{\mu\mu}a - w_0 w_\mu}{b}.$$

□

In the case of $V = \mathbb{C}$ where D_0 acts by Δ_1, Δ_2 from the left and right, respectively, we obtain the following differential operators:

$$\begin{aligned} (1 \otimes K^{\mu} \cdot f)_{\sigma\tau}(u,v) &= (\Delta_2 - \Delta_1 - \alpha_Y(D_0) + 2u\partial_u - (1-u-v)\partial_v) f_{\sigma\tau}(u,v) \\ (1 \otimes K^{\nu} \cdot f)_{\sigma\tau}(u,v) &= (\alpha_Y(F_{\mu}{}^{\nu}) + 2\eta_{\mu\mu} w_0 w_\nu b u \partial_v) f_{\sigma\tau}(u,v) \\ (1 \otimes K^{\rho} \cdot f)_{\sigma\tau}(u,v) &= -\left(\alpha_Y(F^{\rho}{}_{\mu}) + w_\mu w_\nu \frac{\eta_{\mu\mu}a - w_0 w_\mu}{b} \alpha_Y(F^{\rho\nu})\right) f_{\sigma\tau}(u,v). \end{aligned}$$

As a consequence, $\mathcal{D}_2$ corresponds to the following $\mathrm{C}\ell(Y)$-valued differential operator:

$$\begin{aligned} &P_\mu\left(\Delta_2 - \Delta_1 - \frac{d}{2} + 2u\partial_u - (1-u-v)\partial_v\right) \\ &+2\eta_{\mu\mu} w_0 w_\nu P_\nu b u \partial_v \\ &+w_\mu w_\nu \frac{\eta_{\mu\mu}a - w_0 w_\mu}{b} P^{\nu}\left(\alpha_Y(D_0) - \frac{d}{2} - \frac{P_\mu K^{\mu}}{8}\right), \end{aligned}$$

where we don't sum over repeated indices.

5.2.3. As Matrices

Let $f \in \mathrm{Ward}(S \otimes V)$ (for $V = \mathbb{C}$); let's have another look at what values the functions $f_{\sigma\tau}$ ($\sigma, \tau \in \{s,t\}$) assume. From Proposition 4.1.9 and its conclusion, we know that $f_{\sigma\tau}$ can vary freely within $(S \otimes V)^{H_{\sigma\tau}}$ where

$$H_{\sigma\tau} := \mathrm{Stab}(Y_{\sigma\tau}) = \{(g, c_w(g)) \in MA \times MA \mid g \in MA, ge_\mu = e_\mu, ge_\nu = e_\nu\}.$$

Proposition 5.2.9. *The vector space* $(S \otimes V)^{H_{\sigma\tau}}$ *has dimension eight and is spanned by*

$$u_I := \sqrt{2}^{-3\#I} P_{i_1} \wedge \cdots \wedge P_{i_r} \qquad I = \{i_1 < \cdots < i_r\}$$

for $I \subseteq \{\mu, \nu\}$ *or* $\{1, \ldots, n\} \setminus \{\mu, \nu\} \subseteq I \subseteq \{1, \ldots, n\}$.

Proof. Since M acts trivially on V from both sides, the only M-action whose invariants we're looking for, is the one via ϕ_Y on S. As already established in the proof of Lemma 4.3.7, this M-action on S is identical with the adjoint action on Y, extended to the exterior algebra. Consequently, $(S \otimes V)^{H_{\sigma\tau}}$ is closed under taking exterior products. Since $H_{\sigma\tau}$ fixes e_μ, e_ν, we have $u_\mu, u_\nu \in (S \otimes V)^{H_{\sigma\tau}}$.

Furthermore, due to orthogonality, $H_{\sigma\tau}$ leaves the space spanned by $(e_\rho)_{\rho \neq \mu,\nu}$ (the orthogonal complement of e_μ, e_ν) invariant and acts as linear maps of determinant 1. Consequently, for $(g, c_w(g)) \in H_{\sigma\tau}$ we have

$$\phi_Y(g) u_{\{1,\ldots,n\} \setminus \{\mu,\nu\}} = \det{}'(g) u_{\{1,\ldots,n\} \setminus \{\mu,\nu\}} = u_{\{1,\ldots,n\} \setminus \{\mu,\nu\}},$$

where $\det'(g)$ is the determinant of g restricted to the span of $(e_\rho)_{\rho \neq \mu,\nu}$, which is 1. So $u_{\{1,\ldots,n\} \setminus \{\mu,\nu\}}$ is also contained in $(S \otimes V)^{H_{\sigma\tau}}$. This shows that

$$\operatorname{span}\{u_I \mid I \subseteq \{\mu,\nu\} \text{ or } \{1,\ldots,n\} \setminus \{\mu,\nu\} \subseteq I\} \subseteq (S \otimes V)^{H_{\sigma\tau}}.$$

For the opposite inclusion, note that $(S \otimes V)^{H_{\sigma\tau}} \subseteq (S \otimes V)^{\mathfrak{h}_{\sigma\tau}}$. Let $x \in (S \otimes V)^{\mathfrak{h}_{\sigma\tau}}$, say

$$x = \sum_{I \subseteq \{1,\ldots,n\}} a_I u_I.$$

For $1 \leq \alpha, \beta \leq n$ we then have

$$\alpha_Y(F_{\alpha\beta}) x = \sum_{I \subseteq \{1,\ldots,n\}} a_I \alpha_Y(F_{\alpha\beta}) u_I$$

and

$$\alpha_Y(F_{\alpha\beta}) u_I = \begin{cases} 0 & \alpha, \beta \in I \\ (-1)^{\cdots} u_{I \setminus \{\alpha\} \cup \{\beta\}} & \alpha \in I, \beta \notin I \\ (-1)^{\cdots} u_{I \setminus \{\beta\} \cup \{\alpha\}} & \alpha \notin I, \beta \in I \\ 0 & \alpha, \beta \notin I \end{cases}.$$

In other words,

$$\alpha_Y(F_{\alpha\beta}) x = \sum_{I \subseteq \{1,\ldots,n\} \setminus \{\mu,\nu\}} \Big((-1)^{\cdots} a_{I \cup \{\alpha\}} u_{I \cup \{\beta\}} + (-1)^{\cdots} a_{I \cup \{\beta\}} u_{I \cup \{\alpha\}} \Big).$$

Now assume that I is neither a subset of $\{\mu, \nu\}$ nor a superset of $\{1, \ldots, n\}$, then there are α, β with $\{\alpha, \beta\} \cap \{\mu, \nu\} = \emptyset$ such that $\alpha \in I$ and $\beta \notin I$. Then $F_{\alpha\beta} \in \mathfrak{h}_{\sigma\tau}$, hence

$$\begin{aligned} 0 &= \alpha_Y(F_{\alpha\beta}) x \\ &= \sum_{J \subseteq \{1,\ldots,n\} \setminus \{\mu,\nu\}} \Big((-1)^{\cdots} a_{J \cup \{\alpha\}} u_{J \cup \{\beta\}} + (-1)^{\cdots} a_{J \cup \{\beta\}} u_{J \cup \{\alpha\}} \Big) \end{aligned}$$

For all possible choices of J, the vectors $u_{J\cup\{\alpha\}}$ and $u_{J\cup\{\beta\}}$ are linearly independent, hence all $a_{J\cup\{\alpha\}}, a_{J\cup\{\beta\}}$ are zero. In particular, $a_I = a_{I\setminus\{\alpha\}\cup\{\alpha\}} = 0$. This shows that x indeed lies in the span of u_I for $I \subseteq \{\mu,\nu\}$ or $\{1,\ldots,n\} \setminus \{\mu,\nu\} \subseteq I$. □

Note that for this basis we have

$$P_\rho \cdot u_I = \begin{cases} (-1)^{\#\{i\in I, i<\rho\}} 2\sqrt{2} u_{I\cup\{\rho\}} & \rho \notin I \\ 0 & \text{otherwise} \end{cases}$$
$$K^\rho \cdot u_I = \begin{cases} (-1)^{1+\#\{i\in I, i<\rho\}} 2\sqrt{2} u_{I\setminus\{\rho\}} & \rho \in I \\ 0 & \text{otherwise} \end{cases}.$$

For simplicity, introduce the basis $v_I, \tilde{v}_I$ for $I \subseteq \{\mu,\nu\}$ that differs from $u_I, u_{\{1,\ldots,n\}\setminus I}$ only up to signs corresponding to a different ordering of the indices: in our new ordering $\mu \prec \nu$ are smaller than all other possible indices. Then

$$\begin{aligned}
P_\mu : v_\emptyset &\mapsto 2\sqrt{2}v_\mu & v_\nu &\mapsto 2\sqrt{2}v_{\mu\nu} \\
\tilde{v}_{\mu\nu} &\mapsto 2\sqrt{2}\tilde{v}_\nu & \tilde{v}_\mu &\mapsto 2\sqrt{2}\tilde{v}_\emptyset \\
P_\nu : v_\emptyset &\mapsto 2\sqrt{2}v_\nu & v_\mu &\mapsto -2\sqrt{2}v_{\mu\nu} \\
\tilde{v}_{\mu\nu} &\mapsto 2\sqrt{2}\tilde{v}_\mu & \tilde{v}_\nu &\mapsto -2\sqrt{2}\tilde{v}_\emptyset \\
K^\mu : v_\mu &\mapsto -2\sqrt{2}v_\emptyset & v_{\mu\nu} &\mapsto -2\sqrt{2}v_\nu \\
\tilde{v}_\nu &\mapsto -2\sqrt{2}\tilde{v}_{\mu\nu} & \tilde{v}_\emptyset &\mapsto -2\sqrt{2}\tilde{v}_\mu \\
K^\nu : v_\nu &\mapsto -2\sqrt{2}v_\emptyset & v_{\mu\nu} &\mapsto 2\sqrt{2}v_\mu \\
\tilde{v}_\mu &\mapsto -2\sqrt{2}\tilde{v}_{\mu\nu} & \tilde{v}_\emptyset &\mapsto 2\sqrt{2}\tilde{v}_\nu
\end{aligned}$$

and all other elements are mapped to zero. Next, we shall investigate what $K^\nu\Big(\alpha_Y(D_0) + \frac{d}{2} + \frac{K^\mu P_\mu}{8}\Big)$ and $P^\nu\Big(\alpha_Y(D_0) - \frac{d}{2} - \frac{P_\mu K^\mu}{8}\Big)$ do to our basis elements:

v	$\alpha_Y(D_0)v$	$\frac{1}{8}K^\nu K^\mu P_\mu v$	$\frac{1}{8}P^\nu P_\mu K^\mu v$
$v_\emptyset$	$\frac{d}{2}v$	0	0
v_μ	$\frac{d-2}{2}v$	0	$-P^\nu v$
v_ν	$\frac{d-2}{2}v$	$-K^\nu v$	0
$v_{\mu\nu}$	$\frac{d-4}{2}v$	0	0
$\tilde{v}_\emptyset$	$-\frac{d}{2}v$	0	0
$\tilde{v}_\mu$	$-\frac{d-2}{2}v$	$-K^\nu v$	0
$\tilde{v}_\nu$	$-\frac{d-2}{2}v$	0	$-P^\nu v$
$\tilde{v}_{\mu\nu}$	$-\frac{d-4}{2}v$	0	0.

If by ϵ we designate the endomorphism of $(S\otimes V)^{H_{\sigma\tau}}$ mapping

$$v_I \mapsto v_I, \qquad \tilde{v}_I \mapsto -\tilde{v}_I,$$

then we obtain the following actions:

$$K^\nu\left(\alpha_Y(D_0)+\frac{d}{2}+\frac{K^\mu P_\mu}{8}\right)\approx\frac{d-2}{2}(1+\epsilon)K^\nu$$
$$P^\nu\left(\alpha_Y(D_0)-\frac{d}{2}-\frac{P_\mu K^\mu}{8}\right)\approx-\frac{d-2}{2}(1-\epsilon)P^\nu.$$

As a consequence, for $f\in\mathrm{Ward}(S\otimes V)$ we have

$$\begin{aligned}4(\mathcal{D}\cdot f)_{\sigma\tau}(u,v)=&K^\mu\Big(\Delta_1+\Delta_2-\frac{d}{2}+(v+1-u)u\partial_u+(v-1-u)v\partial_v\Big)f_{\sigma\tau}(u,v)\\&-w_\nu K_\nu\Big(2ubw_0(u\partial_u+v\partial_v)-w_\mu\eta_{\nu\nu}\frac{a}{b}\frac{d-2}{2}(1+\epsilon)\Big)f_{\sigma\tau}(u,v)\\&+P_\mu\Big(\Delta_2-\Delta_1-\frac{d}{2}+2u\partial_u-(1-u-v)\partial_v\Big)f_{\sigma\tau}(u,v)\\&+w_\nu\eta_{\mu\mu}P_\nu(2w_0bu\partial_v-w_\nu\eta_{\nu\nu}\frac{a-w_0w_\mu\eta_{\mu\mu}}{b}\frac{d-2}{2}(1-\epsilon))f_{\sigma\tau}(u,v).\end{aligned}$$

If we now define the differential operators

$$\begin{aligned}E&:=-(v+1-u)u\partial_u-(v-1-u)v\partial_v-\Delta_1-\Delta_2+\frac{d}{2}\\F^\pm&:=\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}\Big(-2ub(u\partial_u+v\partial_v)+w_0w_\mu\eta_{\nu\nu}\frac{d-2}{2}(1\pm1)\frac{a}{b}\Big)\\G&:=-2u\partial_u+(1-u-v)\partial_v+\Delta_1-\Delta_2+\frac{d}{2}\\H^\pm&:=\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}\Big(-2ub\partial_v+w_0w_\nu\eta_{\nu\nu}\frac{a-w_0w_\mu\eta_{\mu\mu}}{b}\frac{d-2}{2}(1\pm1)\Big),\end{aligned}$$

we obtain

$$(\mathcal{D}\cdot f)_{\sigma\tau}(u,v)=-\frac{1}{4}\left(K^\mu E+\frac{w_0w_\nu}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}}K^\nu F^\epsilon+P_\mu G+\frac{w_0w_\nu\eta_{\mu\mu}}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}}P_\nu H^{-\epsilon}\right).$$

In the basis $v_\emptyset,v_\mu,v_\nu,v_{\mu\nu},\tilde v_\emptyset,\tilde v_\mu,\tilde v_\nu,\tilde v_{\mu\nu}$ we obtain the following matrix representation:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0&E&fF^\epsilon&0&0&0&0&0\\-G&0&0&-fF^\epsilon&0&0&0&0\\-hH^{-\epsilon}&0&0&E&0&0&0&0\\0&hH^{-\epsilon}&-G&0&0&0&0&0\\0&0&0&0&0&-G&hH^{-\epsilon}&0\\0&0&0&0&E&0&0&-hH^{-\epsilon}\\0&0&0&0&-fF^\epsilon&0&0&-G\\0&0&0&0&0&fF^\epsilon&E&0\end{pmatrix}$$

for

$$f:=\frac{w_0w_\nu}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}},\qquad h:=\frac{w_0w_\nu\eta_{\mu\mu}}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}}$$

(both are 4th roots of unity). Note that if we rearrange our basis to

$$v_\emptyset, v_{\mu\nu}, v_\mu, v_\nu, \tilde{v}_{\mu\nu}, \tilde{v}_\emptyset, \tilde{v}_\nu, \tilde{v}_\mu,$$

we obtain the following matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix}, \qquad D^\pm = \begin{pmatrix} 0 & 0 & E & fF^\pm \\ 0 & 0 & hH^\mp & -G \\ -G & -fF^\pm & 0 & 0 \\ -hH^\mp & E & 0 & 0 \end{pmatrix}. \tag{5.1}$$

So we see that $\mathcal{D}$ leaves invariant the eigenspaces of ϵ. For high enough d ($d > 5$), this is also evident from the fact that any term within $\mathcal{D}$ only changes the degree of an element in S by 1 and leaves $(S \otimes V)^{H_{\sigma\tau}}$ invariant, so in particular the subspaces consisting of degrees $0, 1, 2$ and $d-2, d-1, d$ have to be left invariant as well. Furthermore, since $\mathcal{D}$ is odd, it has to map between the odd and even subspaces of S.

5.3. Other Coordinates

A lot of other coordinates are commonly used in the literature. They are usually valid on a domain locally diffeomorphic to our spaces of cross-ratios. The first set of such coordinates is (z_1, z_2) (usually referred to as $z, \bar{z}$) such that

$$u = \frac{1}{z_1 z_2}, \qquad v = \frac{(z_1 - 1)(z_1 - 2)}{z_1 z_2}.$$

From these equations we can already see that exchanging z_1, z_2 leads to the same u, v. Some algebraic manipulation shows that $\frac{1+u-v}{u} = z_1 + z_2$, so that z_1, z_2 are the roots of the quadric

$$z^2 - \frac{1+u-v}{u} z + \frac{1}{u}.$$

This polynomial has a double root when

$$0 = \left(\frac{1+u-v}{u}\right)^2 - \frac{4}{u} = \frac{1 + u^2 + v^2 - 2u - 2v - 2uv}{u^2},$$

which is the case precisely when $1 + u^2 + v^2 = 2u + 2v + 2uv$, i.e. when our point configuration lies in X_{sl} or X_{tl}, which we've been excluding from our considerations. Inserting the expressions

$$u = \frac{1}{(a - \eta_{\mu\mu} w_0 w_\mu)^2 + \eta_{\mu\mu}\eta_{\nu\nu} b^2}, \qquad v = \frac{a^2 + \eta_{\mu\mu}\eta_{\nu\nu} b^2}{(a - \eta_{\mu\mu} w_0 w_\mu)^2 + \eta_{\mu\mu}\eta_{\nu\nu} b^2}$$

for u, v, we obtain

$$z_{1/2} = 1 - \eta_{\mu\mu} w_0 w_\mu a \pm \sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}} b.$$

In particular, we see that the conditions $b > 0, u, v \neq 0, \infty$ imply that $z_1 \neq z_2$ and $z_1, z_2 \neq 0, 1$.

In this coordinate system, both X_{ss} and X_{tt} (if nonempty) each correspond to a copy of

$$\left\{(z_1, z_2) \in \mathbb{C}^2 \;\middle|\; z_2 = \overline{z_2} \;\middle|\; \mathrm{Im}(z_1) > 0\right\}$$

and X_{st}, X_{ts} each correspond to a copy of

$$\left\{(z_1, z_2) \in \mathbb{R}^2 \setminus \{0, 1\} \;\middle|\; z_1 > z_2\right\}$$

In particular, z_1, z_2 are either both real ($\eta_{\mu\mu}\eta_{\nu\nu} = -1$, then $z_1 > z_2$) or complex conjugates of each other (then z_1 lies in the upper half-plane). In either case, we have

$$2b = \frac{z_1 - z_2}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}}.$$

Next up, we can pick ρ_i such that $\rho_i + \rho_i^{-1} = 2 - 4z_i$, more commonly written as

$$z_i = \frac{1}{2} - \frac{1}{4}(\rho_i + \rho_i^{-1}).$$

Then ρ_i is a root of

$$\rho_i^2 - (2 - 4z_i)\rho_i + 1 = 0,$$

which has two distinct roots unless $z_i(z_i - 1) = 0$, which luckily can never be the case on our domain.

Proposition 5.3.1. *On*

$$Z_c := \{(z_1, z_2) \in \mathbb{C} \mid z_2 = \overline{z_1}, \mathrm{Im}(z_1) > 0\}$$

define

$$\rho_1 = 1 - 2z_1 + 2\sqrt{z_1(z_1 - 1)}, \qquad \rho_2 = 1 - 2z_2 - 2\sqrt{z_2(z_2 - 1)}$$

where $\sqrt{\cdot}$ *has its branch cut along the positive real axis. The two functions* ρ_1, ρ_2 *are well-defined, smooth, and satisfy* $\overline{\rho_1} = \rho_2$.

Proof. We first need to show that neither $z_1(z_1 - 1)$ nor $z_2(z_2 - 1)$ become a positive real number. For $z_i = a + bi$ we have $z_i(z_i - 1) = a(a - 1) - b^2 + (2a - 1)bi$. If this is to be real, we have $2a - 1 = 0$ because $b \neq 0$, i.e. $a = \frac{1}{2}$. Then $z_i(z_i - 1) = -\frac{1}{4} - b^2 < 0$.

This means that we can indeed put our branch cut along the positive real axis, and ρ_1, ρ_2 are well-defined. By the same argument, we could define ρ_1, ρ_2 on the entire open set $\{(z_1, z_2) \in \mathbb{C} \mid \mathrm{Im}(z_1) > 0, \mathrm{Im}(z_2) < 0\}$, where it is a holomorphic function. Thus, its restriction to the real submanifold Z_c is smooth.

Lastly, we have

$$\rho_1^2 - (2 - 4z_1)\rho_1 + 1 = 0.$$

If we conjugate this, we obtain

$$\overline{\rho_1} - (2 - 4z_2)\overline{\rho_1} + 1 = 0,$$

so $\overline{\rho_1}$ is one of the two roots of this equation, hence $\overline{\rho_1} = \rho_2$ or ρ_2^{-1}. We have $\rho_1 - \rho_1^{-1} = 4\sqrt{z_1(z_1-1)}$, which lies in the upper half-plane because that's where the square root maps to. Conversely, $\rho_2 - \rho_2^{-1} = -4\sqrt{z_2(z_2-1)}$ lies in the lower half-plane.

If $\overline{\rho_1} = \rho_2^{-1}$, the difference

$$\rho_2 - \rho_2^{-1} = \overline{\rho_1^{-1} - \rho_1}$$

would be the conjugate of an element of the lower half-plane, hence in the upper half-plane. Since this is not the case, we must have $\rho_2 = \overline{\rho_1}$. □

Note that $\operatorname{Im}(\rho_1 - \rho_1^{-1}) = \operatorname{Im}(\rho)\left(1 + \frac{1}{\|\rho\|^2}\right)$, so ρ_1 also lies in the upper half-plane.

Proposition 5.3.2. *On the domain*

$$Z_r := \left\{(z_1, z_2) \in \mathbb{R}^2 \;\middle|\; z_1 > z_2, z_1, z_2 \notin \{0,1\}\right\}$$

define

$$\rho_i := 1 - 2z_i \pm 2\sqrt{z_i(z_i-1)}$$

(with + for $z_i > 0$ and − for $z_i < 0$) where $\sqrt{\cdot}$ has its branch cut along the negative imaginary axis. Then ρ_1, ρ_2 are well-defined and smooth. Then ρ_1, ρ_2 both map to the union of the intervals $(-1,0) \cup (0,1)$ and the upper half-circle of radius 1. If arg *is the argument function with branch cut along the negative imaginary axis, then* $\arg(\rho_1 - \frac{1}{2}i) > \arg(\rho_2 - \frac{1}{2}i)$.

Proof. When $z_i \in \mathbb{R}$, then so is $z_i(z_i - 1)$, so the functions ρ_1, ρ_2 with the square root as specified is well-defined. For smoothness it suffices to check that ρ_i depends smoothly on z_i. Since z_i can attain any real value except for $0, 1$, it suffices to check on the three intervals $(-\infty, 0), (0,1), (1,\infty)$. On all of these intervals, ρ_i equals

$$f_1(z_i) := 1 - 2z_i + 2\sqrt{z_i(z_i-1)} \quad \text{or} \quad f_2(z_i) := 1 - 2z_i - 2\sqrt{z_i(z_i-1)}.$$

Both of these functions can be defined in an open neighbourhood of the real axis to be holomorphic functions. Then their restrictions to the real intervals are at least smooth. We have

$$\lim_{z_i \to 0} \rho_i = 1, \qquad \lim_{z_i \to -\infty} \rho_i = \lim_{z_i \to \infty} \rho_i = 0, \lim_{z_i \to 1} \rho_i = -1,$$

and $\rho_i \in \mathbb{R}$ for $z_i^2 > z_i$, i.e. for $|z_i| > 1$. Together with the limits, this shows that $(-\infty, 0)$ is mapped to $(0,1)$ and $(1,\infty)$ is mapped to $(-1,0)$. Furthermore, since the derivative has no roots, these mappings are strictly monotonic (increasing in both cases). For the interval $(0,1)$, note that $z_i^2 < z_i$, so that the square root becomes imaginary. In particular,

$$\|\rho_i\|^2 = (1 - 2z_i)^2 - 4z_i(z_i - 1) = 1,$$

with $\operatorname{Im}(\rho_i) = 2\sqrt{z_i(1-z_i)} > 0$. This shows that the interval $(0,1)$ is mapped to the upper half-circle

$$\{z \in \mathbb{C} \mid |z| = 1, \operatorname{Im}(z) > 0\}.$$

All in all, ρ_i's graph is a semicircle with three punctures that is being traversed anticlockwise exactly once. Thus, we can compare z_1, z_2 by comparing how "far along the semicircle" ρ_1, ρ_2 are when compared to each other.

To make this last argument rigorous, note that

$$f'_{1/2}(z) = -2 \pm \frac{2z-1}{\sqrt{z(z-1)}} = \pm \frac{2z - 1 \mp 2\sqrt{z(z-1)}}{\sqrt{z(z-1)}},$$

which has a root only if $2z - 1 = \pm 2\sqrt{z(z-1)}$. If this is the case, then $(2z-1)^2 = 4z^2 - 4z + 1 = 4z^2 - 4z$, i.e. $1 = 0$, which is a contradiction. Thus, neither f_1 nor f_2 have critical points. As a consequence, the function

$$\mathbb{R}\backslash \to (0, 2\pi), \qquad t \mapsto \arg(\rho_i(t) - \frac{1}{2}i)$$

(mapping 0 to $\arg(1 - \frac{1}{2}i)$ and 1 to $\arg(-1 - \frac{1}{2}i)$), which is continuous on $\mathbb{R}$ and smooth on $\mathbb{R} \setminus \{0, 1\}$, must be monotonic on each of its three intervals of differentiability. Furthermore, the derivative is positive on all three intervals, which implies the claim. $\square$

With exactly these choices of square roots, we defined ρ_1, ρ_2 such that

$$2b = \frac{z_1 - z_2}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}}.$$

Lastly, introduce χ_1, χ_2 with

$$\exp(\chi_1) = \rho_i.$$

Then Z_c corresponds to

$$\left\{(\chi_1, \chi_2) \in \mathbb{C} \;\middle|\; \chi_2 = \overline{\chi_1}, 0 < \mathrm{Im}(\chi_1) < \pi, \mathrm{Im}(\sinh^2(\chi_1/2)) < 0\right\}.$$

Using that $\mathrm{Im}(\sinh^2(\chi_1/2)) = \frac{1}{2}\sinh(\mathrm{Re}(\chi))\sin(\mathrm{Im}(\chi))$, this equals

$$\{(\chi_1, \chi_2) \in \mathbb{C} \mid \chi_2 = \overline{\chi_1}, 0 < \mathrm{Im}(\chi_1) < \pi, \mathrm{Re}(\chi_1) < 0\}.$$

From here, we can in particular also gather that in (ρ_1, ρ_2)-coordinates, the domain Z_c is exactly

$$\{(\rho_1, \rho_2) \in \mathbb{C} \mid \rho_2 = \overline{\rho_1}, \mathrm{Im}(\chi_1), \mathrm{Re}(\chi_1) > 0\}.$$

Similarly, in (χ_1, χ_2)-coordinates, Z_r corresponds to

$$\{(\chi_1, \chi_2) \in (\mathbb{R}_{<0} + \{0, i\pi\}) \cup \{i(0, \pi)\} \mid \chi_1 \text{ "further along than" } \chi_2\}.$$

Thus, we find that in (χ_1, χ_2)-coordinates, either χ_1 is confined to be inside the strip $\mathbb{R}_{<0} + (0, \pi)i$ and then χ_2 is its complex conjugate, or both are confined to the strips boundary, where – seen from $-\infty$ an measured in anticlockwise direction, χ_1 is "further along" the boundary than χ_2.

5.3.1. Differential Operators

Proposition 5.3.3. *We have*

$$u\partial_u = \frac{1}{z_1 - z_2}(z_1(1-z_1)\partial_{z_1} - z_2(1-z_2)\partial_{z_2})$$
$$v\partial_v = -\frac{(1-z_1)(1-z_2)}{z_1 - z_2}(z_1\partial_{z_1} - z_2\partial_{z_2}).$$

Proof. Starting from $u = (z_1 z_2)^{-1}$ and $v/u = (1-z_1)(1-z_2)$, we have

$$\frac{\partial u}{\partial z_i} = -\frac{u}{z_i}, \qquad \frac{\partial v}{\partial z_i} = \frac{v}{z_i(z_i - 1)}.$$

Thus,

$$\begin{aligned}\partial_{z_i} &= \frac{\partial u}{\partial z_i}\partial_u + \frac{\partial v}{\partial z_i}\partial_v \\ &= -\frac{u}{z_i}\partial_u + \frac{v}{z_i(z_i-1)}\partial_v \\ z_i\partial_{z_i} &= -u\partial_u + \frac{1}{z_i - 1}v\partial_v.\end{aligned}$$

Solving this 2×2 system of linear equations yields

$$\begin{pmatrix} u\partial_u \\ v\partial_v \end{pmatrix} = \frac{1}{z_2 - z_1}\begin{pmatrix} z_1 - 1 & 1 - z_2 \\ (z_1-1)(z_2-1) & -(z_1-1)(z_2-1) \end{pmatrix}\begin{pmatrix} z_1\partial_{z_1} \\ z_2\partial_{z_2} \end{pmatrix},$$

as claimed. □

As a consequence, we have

$$\begin{aligned}(1-u-v)\partial_v &= \frac{1-u-v}{v}v\partial_v = \frac{2-z_1-z_2}{z_1-z_2}(z_1\partial_{z_1} - z_2\partial_{z_2}) \\ u\partial_u + v\partial_v &= \frac{1}{z_1-z_2}(z_1(1-z_1)(1-(1-z_2))\partial_{z_1} - z_2(1-z_2)(1-(1-z_1))\partial_{z_2}) \\ &= -\frac{z_1 z_2}{z_1 - z_2}((z_1-1)\partial_{z_1} - (z_2-1)\partial_{z_2}).\end{aligned}$$

Proposition 5.3.4. *For $i = 1, 2$ we have*

$$4\frac{\rho_i\partial_{\rho_i}}{\rho_i - \rho_i^{-1}} = -\partial_{z_i}.$$

Proof. Recall that $4z_i = 2 - \rho_i - \rho_i^{-1}$. Differentiating this, we obtain

$$\frac{\partial z_i}{\partial \rho_i} = \frac{1}{4}(\rho_i^{-2} - 1),$$

hence

$$4\rho_i\partial_{\rho_i} = \rho_i(\rho_i^{-2} - 1)\partial_{z_i} = -(\rho_i - \rho_i^{-1})\partial_{z_i}.$$

Multiplying through by $\rho_i - \rho_i^{-1}$ yields the claim. □

From this we can conclude that

$$(\rho_i\partial_{\rho_i})^2 = -\frac{1}{4}\rho_i\partial_{\rho_i}(\rho_i - \rho_i^{-1})\partial_{z_i} = -\frac{1}{4}(\rho_i - \rho_i^{-1})\rho_i\partial_{\rho_i}\partial_{z_i} - \frac{1}{4}(\rho_i + \rho_i^{-1})\partial_{z_i}$$
$$= \frac{1}{16}(\rho_i - \rho_i^{-1})^2\partial_{z_i}^2 - \left(\frac{1}{2} - z_i\right)\partial_{z_i} = -z_i(1-z_i)\partial_{z_i}^2 - \left(\frac{1}{2} - z_i\right)\partial_{z_i}.$$

Proposition 5.3.5. *Lastly, we have*

$$\partial_{\chi_i} = \rho_i\partial_{\rho_i}.$$

Proof. Recall that $\rho_i = \exp(\chi_i)$, then

$$\partial_{\chi_i} = \frac{\partial\rho_i}{\partial\chi_i}\partial_{\rho_i} = \rho_i\partial_{\rho_i}$$ □

5.3.2. Casimir Operator

The differential operator from Corollary 5.1.4 then becomes

$$-\frac{2-z_1-z_2}{z_1-z_2}(z_1\partial_{z_1} - z_2\partial_{z_2})\left(\frac{(1-z_1)(1-z_2)}{z_1z_2}(z_1\partial_{z_1} - z_2\partial_{z_2}) - \frac{\Delta_1+\Delta_2}{2}\right)$$
$$+\frac{2}{z_1-z_2}(z_1(1-z_1)\partial_{z_1} - z_2(1-z_2)\partial_{z_2})\left(\frac{1}{z_1-z_2}(z_1(1-z_1)\partial_{z_1} - z_2(1-z_2)\partial_{z_2}) + \Delta_2 - \frac{d}{2}\right)$$
$$+\frac{z_1+z_2}{z_1-z_2}((z_1-1)\partial_{z_1} - (z_2-1)\partial_{z_2})\left(-\frac{z_1z_2}{z_1-z_2}((z_1-1)\partial_{z_1} - (z_2-1)\partial_{z_2}) - \frac{\Delta_1-\Delta_2}{2}\right)$$
$$+\frac{\Delta_2^2 - d\Delta_2}{2},$$

which equals

$$-z_1(1-z_1)\partial_{z_1}^2 - \left(\frac{\Delta_1-\Delta_2}{2} + 1 - (1-\Delta_2+1)z_1\right)\partial_{z_1}$$
$$-z_2(1-z_2)\partial_{z_2}^2 - \left(\frac{\Delta_1-\Delta_2}{2} + 1 - (1-\Delta_2+1)z_2\right)\partial_{z_2}$$
$$-\frac{d-2}{z_1-z_2}(z_1(1-z_1)\partial_{z_1} - z_2(1-z_2)\partial_{z_2})$$
$$+\frac{\Delta_2(\Delta_2-d)}{2}.$$

Next, we move to (ρ_1, ρ_2) coordinates. Here we get

$$\begin{aligned}
&(\rho_1\partial_{\rho_1})^2 - \left(\frac{\Delta_1 - \Delta_2 + 1}{2} - (1-\Delta_2)z_1\right)\partial_{z_1} \\
&+(\rho_2\partial_{\rho_2})^2 - \left(\frac{\Delta_1 - \Delta_2 + 1}{2} - (1-\Delta_2)z_2\right)\partial_{z_2} \\
&-\frac{d-2}{z_1 - z_2}(z_1(1-z_1)\partial_{z_1} - z_2(1-z_2)\partial_{z_2}) + \frac{\Delta_2(\Delta_2 - d)}{2} \\
&=(\rho_1\partial_{\rho_1})^2 + (2\Delta_1 + (1-\Delta_2)(\rho_1 + \rho_1^{-1}))\frac{\rho_1\partial_{\rho_1}}{\rho_1 - \rho_1^{-1}} \\
&+(1 \leftrightarrow 2) \\
&+\frac{d-2}{\rho_1 + \rho_1^{-1} - \rho_2 - \rho_2^{-1}}\Big((\rho_1 - \rho_1^{-1})\rho_1\partial_{\rho_1} - (\rho_2 - \rho_2)^{-1}\rho_2\partial_{\rho_2}\Big) \\
&+\frac{\Delta_2(\Delta_2 - d)}{2} \\
&=(\rho_1\partial_{\rho_1})^2 + \left(\Delta_1\frac{1+\rho_1^{-1}}{1-\rho_1^{-1}} + (1-\Delta_1-\Delta_2)\frac{1+\rho_1^{-2}}{1-\rho_1^{-2}}\right)\rho_1\partial_{\rho_1} \\
&+(1 \leftrightarrow 2) \\
&+\frac{d-2}{2}\left(\frac{1+\rho_1^{-1}\rho_2}{1-\rho_1^{-1}\rho_2}(\rho_1\partial_{\rho_1} - \rho_2\partial_{\rho_2}) + \frac{1+(\rho_1\rho_2)^{-1}}{1-(\rho_1\rho_2)^{-1}}(\rho_1\partial_{\rho_1} + \rho_2\partial_{\rho_2})\right) \\
&+\frac{\Delta_2(\Delta_2 - d)}{2}.
\end{aligned}$$

Lastly, for (χ_1, χ_2)-coordinates note that the operator already only contains expressions of the shape $\rho_i\partial_{\rho_i}$, so it becomes easy to translate the operator:

$$\begin{aligned}
&\partial_{\chi_1}^2 + \partial_{\chi_2}^2 + \left(\frac{1-\Delta_2}{2}\right)^2 + \left(\frac{d-\Delta_2-1}{2}\right)^2 - \frac{1}{2^2} - \left(\frac{d-1}{2}\right)^2 \\
&+\left(\Delta_1 \coth\left(\frac{\chi_1}{2}\right) + (1-\Delta_1-\Delta_2)\coth(\chi_1)\right)\partial_{\chi_1} \\
&+\left(\Delta_1 \coth\left(\frac{\chi_2}{2}\right) + (1-\Delta_1-\Delta_2)\coth(\chi_2)\right)\partial_{\chi_2} \\
&+\frac{d-2}{2}\left(\coth\left(\frac{\chi_1-\chi_2}{2}\right)(\partial_{\chi_1} - \partial_{\chi_2}) + \coth\left(\frac{\chi_1+\chi_2}{2}\right)(\partial_{\chi_1} + \partial_{\chi_2})\right)
\end{aligned}$$

Define

$$k \in \mathbb{C}^3, \qquad k = \left(\Delta_1, \frac{d-2}{2}, \frac{1-\Delta_1-\Delta_2}{2}\right).$$

Then this becomes

$$\begin{aligned}
&\partial_{\chi_1}^2 + \partial_{\chi_2}^2 + \left(\frac{k_1}{2} + k_3\right)^2 + \left(\frac{k_1}{2} + k_2 + k_3\right)^2 - \frac{1}{2^2} - \left(k_2 + \frac{1}{2}\right)^2 \\
&+ \left(k_1 \coth\left(\frac{\chi_1}{2}\right) + 2k_3 \coth(\chi_1)\right)\partial_{\chi_1} + \left(k_1 \coth\left(\frac{\chi_2}{2}\right) + 2k_3 \coth(\chi_2)\right)\partial_{\chi_2} \\
&+ k_2\left(\coth\left(\frac{\chi_1 - \chi_2}{2}\right)(\partial_{\chi_1} - \partial_{\chi_2}) + \coth\left(\frac{\chi_1 + \chi_2}{2}\right)(\partial_{\chi_1} + \partial_{\chi_2})\right).
\end{aligned} \tag{5.2}$$

5.3.3. Dirac Operator

We will now represent the differential operators $E, F^{\pm}, G, H^{\pm}$ in the different coordinates.

We have

$$\begin{aligned}
E &= -\left(\frac{(1-z_1)(1-z_2)}{z_1 z_2} + 1 - \frac{1}{z_1 z_2}\right)\frac{1}{z_1 - z_2}(z_1(1-z_1)\partial_{z_1} - z_2(1-z_2)\partial_{z_2}) \\
&\quad + \left(\frac{(1-z_1)(1-z_2)}{z_1 z_2} - 1 - \frac{1}{z_1 z_2}\right)\frac{(1-z_1)(1-z_2)}{z_1 z_2}(z_1\partial_{z_1} - z_2\partial_{z_2}) - \Delta_1 - \Delta_2 + \frac{d}{2} \\
&= (z_1 - 1)\partial_{z_1} + (z_2 - 1)\partial_{z_2} - \Delta_1 - \Delta_2 + \frac{d}{2} \\
&= (z_1 - 1)\partial_{z_1} + (z_2 - 1)\partial_{z_2} + k_2 + 2k_3 \\
G &= -\frac{2}{z_1 - z_2}(z_1(1-z_1)\partial_{z_1} - z_2(1-z_2)\partial_{z_2}) + \frac{2 - z_1 - z_2}{z_1 - z_2}(z_1\partial_{z_1} - z_2\partial_{z_2}) + \Delta_1 - \Delta_2 + \frac{d}{2} \\
&= \frac{1}{z_1 - z_2}((z_1 - z_2)z_1\partial_{z_1} - (z_2 - z_1)z_2\partial_{z_2}) + \Delta_1 - \Delta_2 + \frac{d}{2} \\
&= z_1\partial_{z_1} + z_2\partial_{z_2} + 2k_1 + k_2 + 2k_3.
\end{aligned}$$

For $F^{\pm}, H^{\pm}$, recall that

$$2b = \frac{z_1 - z_2}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}},$$

so that

$$\begin{aligned}
\frac{a}{b} &= -\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}\,\eta_{\mu\mu} w_0 w_\mu \frac{z_1 + z_2 - 2}{z_1 - z_2} \\
\frac{a - w_0 w_\mu \eta_{\mu\mu}}{b} &= -w_0 w_\mu \eta_{\mu\mu}\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}\frac{z_1 + z_2}{z_1 - z_2}.
\end{aligned}$$

This gives us

$$\begin{aligned}
F^{\pm} &= \frac{1}{z_1 z_2}(z_1 - z_2)\frac{z_1 z_2}{z_1 - z_2}((z_1 - 1)\partial_{z_1} - (z_2 - 1)\partial_{z_2}) - \eta_{\mu\mu}\eta_{\nu\nu}\frac{d-2}{2}(1 \pm 1)(-\eta_{\mu\mu}\eta_{\nu\nu})\frac{z_1 + z_2 - 2}{z_1 - z_2} \\
&= (z_1 - 1)\partial_{z_1} - (z_2 - 1)\partial_{z_2} + (1 \pm 1)k_2\frac{z_1 + z_2 - 2}{z_1 - z_2} \\
H^{\pm} &= (z_1 - z_2)\frac{1}{z_1 - z_2}(z_1\partial_{z_1} - z_2\partial_{z_2}) - w_\mu w_\nu \eta_{\nu\nu}\eta_{\mu\mu}(-\eta_{\mu\mu}\eta_{\nu\nu})\frac{d-2}{2}(1 \pm 1)\frac{z_1 + z_2}{z_1 - z_2} \\
&= z_1\partial_{z_1} - z_2\partial_{z_2} + w_\mu w_\nu(1 \pm 1)k_2\frac{z_1 + z_2}{z_1 - z_2}.
\end{aligned}$$

Now, we go to (ρ_1, ρ_2)-coordinates. For that, note that

$$\begin{aligned} z_i \partial_{z_i} &= (-2 + \rho_i + \rho_i^{-1}) \frac{\rho_i \partial_{\rho_i}}{\rho_i - \rho_i^{-1}} \\ &= \frac{1 - \rho_i}{1 + \rho_i} \rho_i \partial_{\rho_i} = -\frac{1 - \rho_i^{-1}}{1 + \rho_i^{-1}} \rho_i \partial_{\rho_i} \\ (z_i - 1)\partial_{z_i} &= (2 + \rho_i + \rho_i^{-1}) \frac{\rho_i \partial_{\rho_i}}{\rho_i - \rho_i^{-1}} \\ &= \frac{1 + \rho_i^{-1}}{1 - \rho_i^{-1}} \rho_i \partial_{\rho_i}. \end{aligned}$$

Thus,

$$\begin{aligned} E &= \frac{1 + \rho_1^{-1}}{1 - \rho_1^{-1}} \rho_1 \partial_{\rho_1} + \frac{1 + \rho_2^{-1}}{1 - \rho_2^{-1}} \rho_2 \partial_{\rho_2} + k_2 + 2k_3 \\ G &= -\frac{1 - \rho_1^{-1}}{1 + \rho_1^{-1}} \rho_1 \partial_{\rho_1} - \frac{1 - \rho_2^{-1}}{1 + \rho_2^{-1}} \rho_2 \partial_{\rho_2} + 2k_1 + k_2 + 2k_3 \\ F^{\pm} &= \frac{1 + \rho_1^{-1}}{1 - \rho_1^{-1}} \rho_1 \partial_{\rho_1} - \frac{1 + \rho_2^{-1}}{1 - \rho_2^{-1}} \rho_2 \partial_{\rho_2} + (1 \pm 1) k_2 \frac{(1 + \rho_1)(1 + \rho_1^{-1}) + (1 + \rho_2)(1 + \rho_2^{-1})}{(\rho_1 - \rho_2)(1 - (\rho_1 \rho_2)^{-1})} \\ H^{\pm} &= -\frac{1 - \rho_1^{-1}}{1 + \rho_1^{-1}} \rho_1 \partial_{\rho_1} + \frac{1 - \rho_2^{-1}}{1 + \rho_2^{-1}} \rho_2 \partial_{\rho_2} - w_\mu w_\nu (1 \pm 1) k_2 \frac{(1 - \rho_1)(1 - \rho_1^{-1}) + (1 - \rho_2)(1 - \rho_2^{-1})}{(\rho_1 - \rho_2)(1 - (\rho_1 \rho_2)^{-1})} \end{aligned}$$

and in (χ_1, χ_2) coordinates

$$\begin{aligned} E &= \coth\left(\frac{\chi_1}{2}\right) \partial_{\chi_1} + \coth\left(\frac{\chi_2}{2}\right) \partial_{\chi_2} + k_2 + 2k_3 \\ G &= -\tanh\left(\frac{\chi_1}{2}\right) \partial_{\chi_1} - \tanh\left(\frac{\chi_1}{2}\right) \partial_{\chi_2} + 2k_1 + k_2 + 2k_3 \\ F^{\pm} &= \coth\left(\frac{\chi_1}{2}\right) \partial_{\chi_1} - \coth\left(\frac{\chi_2}{2}\right) \partial_{\chi_2} + (1 \pm 1) k_2 \frac{\cosh^2\left(\frac{\chi_1}{2}\right) + \cosh^2\left(\frac{\chi_2}{2}\right)}{\sinh\left(\frac{\chi_1 - \chi_2}{2}\right) \sinh\left(\frac{\chi_1 + \chi_2}{2}\right)} \\ H^{\pm} &= -\tanh\left(\frac{\chi_1}{2}\right) \partial_{\chi_1} + \tanh\left(\frac{\chi_1}{2}\right) \partial_{\chi_2} - w_\mu w_\nu (1 \pm 1) k_2 \frac{\sinh^2\left(\frac{\chi_1}{2}\right) + \sinh^2\left(\frac{\chi_2}{2}\right)}{\sinh\left(\frac{\chi_1 - \chi_2}{2}\right) \sinh\left(\frac{\chi_1 + \chi_2}{2}\right)} \end{aligned}$$

6. Dunkl Operators

Let us now shift gears and talk about the theory developed around the CS model. Recall from the introduction that everything is phrased in terms of root system, so we had best begin by recalling some basic facts about (abstract) root systems. A more thorough introduction to root systems and in particular their classification (which we won't touch on) can be found at [Bou68, chap. VI] and [Hum72, part III]. (Note that [Hum72] restricts himself to reduced root systems, which we will not.)

6.1. Root Systems

Notation 6.1.1. *Let $\mathfrak{a}$ be a finite-dimensional real vector space with (positive-definite) inner product.*

(a) For any element $\alpha \in \mathfrak{a}^*$ write $X_\alpha \in \mathfrak{a}$ for the unique element such that $\alpha(Y) = \langle X_\alpha, Y \rangle$ for all $y \in \mathfrak{a}$.

(b) For $\alpha \in \mathfrak{a}^*$ write

$$\alpha^\vee := \frac{2X_\alpha}{\|X_\alpha\|^2}.$$

(c) For $\alpha \in \mathfrak{a}^* \setminus \{0\}$ define $r_\alpha \in GL(\mathfrak{a}^*)$ by $r_\alpha(\beta) := \beta - \beta(\alpha^\vee)\alpha$. As an abuse of notation, also define $r_\alpha \in GL(\mathfrak{a})$ by $r_\alpha(X) := X - \alpha(X)\alpha^\vee$. This is the reflection *along α.*

Definition 6.1.2. *(a) A finite subset $R \subseteq \mathfrak{a}^* \setminus \{0\}$ is called an* (abstract) root system *if*

a) R spans $\mathfrak{a}^*$.

b) For each $\alpha \in R$ we have $r_\alpha(R) = R$.

c) For each $\alpha, \beta \in R$ we have $\alpha(\beta^\vee) \in \mathbb{Z}$.

The elements $\alpha \in R$ are then called roots*, and the $\alpha^\vee$ are called* coroots*. The set $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ is also an abstract root system.*

(b) The sets

$$R^0 := R \setminus \frac{1}{2}R, \qquad R_0 := R \setminus 2R$$

of inmultipliable *and* indivisible *roots, respectively, are also root systems.*

(c) The group $W \leq GL(\mathfrak{a}^*)$ generated by the r_α ($\alpha \in R$) is called the Weyl group*.*

(d) *Write* $\mathbb{Z}R$ *and* $\mathbb{Z}R^\vee$ *for the lattices generated by* $R, R^\vee$*. They are called the* root *and* coroot lattices*, respectively.*

(e) *The set*

$$P := \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}R^\vee, \mathbb{Z}) = \{\lambda \in \mathfrak{a}^* \mid \forall \alpha \in R : \lambda(\alpha^\vee) \in \mathbb{Z}\}$$

is called the weight lattice*.*

(f) *Define*

$$H := \mathrm{Hom}_\mathbb{Z}(P, \mathbb{C}^\times)$$

to be the complex torus dual to P*.*

Lemma 6.1.3. *R and R_0 have the same root lattice, R and R^0 have the same coroot lattice and hence the same weight lattice and complex torus H.*

Proof. Because of $(R^0)^\vee = (R^\vee)_0$, we only have to show that R and R_0 have the same root lattice. This follows from $R \subseteq \mathbb{Z}R_0$. □

Definition 6.1.4. *A subset* $R^+ \subseteq R$ *is called a* set of positive roots *if*

(a) $R = R^+ \sqcup (-R^+)$ *(write* $R^- := -R^+$*).*

(b) 0 *is not contained in the convex hull of* R^+*.*

Write $R^{0,\pm}$ *and* $R_0^\pm$ *for* $R^0 \cap R^\pm$ *and* $R_0 \cap R^\pm$*.*
A root $\alpha \in R^+$ *is called* simple *if for every way of writing it as*

$$\alpha = \sum_{i=1}^{r} z_i \beta_i$$

for $\beta_i \in R^+$ *and* $z_i \neq 0$ *we have* $r = 1$*,* $\beta_1 = \alpha$*, and* $z_1 = 1$*.*

A set of positive roots of R then also gives a set of positive roots of $R^\vee$, and by considering the cones generated by $R^+, R^{\vee+}$ also orders on $\mathbb{Z}R, \mathbb{Z}R^\vee$. A weight μ is called *dominant* if $\mu((\mathbb{Z}R^\vee)^+) \subseteq \mathbb{R}_{\geq 0}$, write P^+ for the set of dominant weights.

Lemma 6.1.5. *For every root system there exists a set of positive roots.*

Proof. Let $X \in \mathfrak{a}$ so that no root $\alpha \in R$ satisfies $\alpha(X) = 0$, then

$$R^+ := \{\alpha \in R \mid \alpha(X) > 0\}$$

does the job. Such an X exists for the following reason: for $\alpha \in R$ write $H_\alpha := \ker(\alpha)$. Then $\mathfrak{a} \setminus L_\alpha$ is open and dense in $\mathfrak{a}$. Since $\mathfrak{a}$ is a finite-dimensional vector space, it is a complete metric space, so by the Baire category theorem, the (even finite!) intersection

$$\{X \in \mathfrak{a} : \forall \alpha \in R : \alpha(X) \neq 0\} = \bigcap_{\alpha \in R} (\mathfrak{a} \setminus H_\alpha)$$

is dense, hence nonempty. □

In particular, since the (closures of the) convex hulls of R^+ and R^- are supposed not to intersect and are bounded, the separating hyperplane theorem tells us that every set of positive roots can be found this way.

Theorem 6.1.6. *S is a basis for $\mathfrak{a}^*$, and every root $\alpha \in R$ can be expanded in terms of S with either only nonnegative or only nonpositive coefficients.*

Proof. See [Hum72, section 10.1, theorem']. □

Lemma 6.1.7. *Let $\alpha \in S$, then r_α permutes the positive roots that are not multiples of α. In other words:*

$$r_\alpha(R^+) = R^+ \setminus \mathbb{R}\alpha \cup \left\{-c\alpha \,\middle|\, c\alpha \in R^+\right\}.$$

Proof. See [Bou68, chap. VI, §1, no. 6, cor. 1]. □

Let $\alpha_1, \dots, \alpha_n \in R^0$ be simple roots, then so are $\alpha_1^\vee, \dots, \alpha_n^\vee \in R^\vee$. This means that they are a basis of the coroot lattice. Define $\omega_i(\alpha_j^\vee) := \delta_{ij}$. They form a basis of the lattice dual to $\mathbb{Z}R^\vee$ (the lattice of weights) and are called the *fundamental weights.* Furthermore, define $r_i := r_{\alpha_i}$, then W is a finite Coxeter group generated by $r_1, \dots, r_n$ (see [Bou68, chap. VI, §1, no. 5]).

To H we associate the polynomial ring $\mathbb{C}[H]$ that is the ring generated by functions of the shape $e(\lambda)$ ($\lambda \in P$). Here $e(\lambda) : H \to \mathbb{C}, \phi \mapsto \phi(\lambda)$. They satisfy $e(\lambda)e(\mu) = e(\lambda+\mu)$, and evidently, $\mathbb{C}[H] = \mathbb{C}[e(\omega_1), \dots, e(\omega_n)]$. Note that via the map

$$H \to (\mathbb{C}^\times)^n, \qquad \phi \mapsto (\phi(\omega_1), \dots, \phi(\omega_n)),$$

H is a complex manifold of dimension n and it therefore makes sense to not only speak about polynomials on H (or localisations thereof), but also of holomorphic functions. Note that R's linear action on $\mathfrak{a}^*$ induces an action on $\mathfrak{a}$, and together they leave $R^\vee, \mathbb{Z}R, \mathbb{Z}R^\vee, P, \mathbb{C}[H]$ invariant. In particular, if $U \subseteq H$ is an open subset invariant under W, then $\mathcal{O}(U)$, the space of holomorphic functions on U is also acted upon by W.

Define

$$H^{reg} := \{\phi \in H \mid \forall \alpha \in R : \phi(\alpha) \neq 1\} = \{\phi \in H \mid \forall w \in W : w(\phi) \neq \phi\}$$

to be the set of regular points of H under the action of W. This is an open subvariety whose coordinate ring $\mathbb{C}[H^{reg}]$ is the localisation of $\mathbb{C}[H]$ in the multiplicative set containing all $1 - e(\alpha)$ ($\alpha \in R$). For details about $\mathbb{C}[H]$, consult [Bou68, chap. VI, §3].

6.2. Differential Reflection Operators

The Dunkl operators are going to be (mildly singular) differential reflection operators. What that means is what we're going to explore in this section. Fix a root system R, a set of positive roots R^+, and write $\mathfrak{h} = \mathfrak{a} \otimes \mathbb{C}$ for the complexification of R's carrier space.

The three keywords here are "singular", "differential", and "reflection". We're going to address each of them in turn:

Definition 6.2.1. *(a) Define $\mathfrak{R} \subseteq \mathbb{C}(H)$ (the fraction field of $\mathbb{C}[H]$) to be the subalgebra generated by 1 and $(1 - e(-\alpha))^{-1}$ ($\alpha \in R$).*

(b) Write $S(\mathfrak{h})$ for the symmetric (tensor) algebra of $\mathfrak{h}$, interpreted as polynomial functions on $\mathfrak{h}^*$. Write the elements as ∂_p (for $p \in S(\mathfrak{h})$).

(c) Write $\mathbb{C}[W]$ for the group algebra of W.

We ultimately want to turn their tensor product into an algebra. For that we proceed stepwise.

Proposition 6.2.2. *$S(\mathfrak{h})$ acts on $\mathfrak{R}$ by means of derivations.*

Proof. $S(\mathfrak{h})$ acts on $\mathbb{C}[H]$ via $\partial_p e(\lambda) = p(\lambda)e(\lambda)$ ($\lambda \in P$), which is a (higher-order) derivation. This can be extended to $\mathbb{C}(H)$, so we just need to show that $\mathfrak{R}$ is left invariant.

Let $\xi \in \mathfrak{h}$ and $\alpha \in R$, then

$$\begin{aligned}\partial_\xi \frac{1}{1 - e(-\alpha)} &= \alpha(\xi) \frac{-e(-\alpha)}{(1 - e(-\alpha))^2} \\ &= \alpha(\xi) \Big(1 - \frac{1}{1 - e(-\alpha)}\Big) \frac{1}{1 - e(-\alpha)} \in \mathfrak{R}.\end{aligned}$$

□

Using this action, we can iterate the construction from [Bou58, chap. VIII, §1, no. 4, prop. 7] (using $\sigma = \mathrm{id}$, $d = \partial_{\xi_i}$, for a basis $\xi_1, \ldots, \xi_n \in \mathfrak{h}$) to obtain an algebra structure on $\mathfrak{R} \otimes S(\mathfrak{h})$ that exactly captures the differential operator behaviour. Call this algebra $\mathbb{D}_{\mathfrak{R}}$, the algebra of $\mathfrak{R}$-valued differential operators.

Proposition 6.2.3. *W acts on $\mathbb{D}_{\mathfrak{R}}$ by algebra automorphisms.*

Proof. Note that W leaves R invariant, hence $(1 - e(-\alpha))^{-1}$ for any $\alpha \in R$ is mapped to $\mathfrak{R}$ again. This shows that W acts both on $\mathfrak{R}$ and on $S(\mathfrak{h})$ by algebra automorphisms, so let's show that the same holds for $\mathbb{D}_{\mathfrak{R}}$. By expanding in terms of a basis of $\mathfrak{h}$, separating degrees, and induction the question reduces to

$$w \cdot (\partial_\xi f) = (w \cdot \partial_\xi)(w \cdot f)$$

for $f \in \mathfrak{R}$ and $\xi \in \mathfrak{h}$ (for $f\xi$ it is evidently true since $f\partial_\xi = f \otimes \xi$, so it follows from definition).

We have

$$w \cdot (\partial_\xi f) = w \cdot (\partial_\xi(f) \otimes 1) + w \cdot (f \otimes \xi) = (\partial_{w(\xi)} \cdot (w \cdot f)) \otimes 1 + (w \cdot f) \otimes w(\xi) = \partial_{w(\xi)} w \cdot f.$$ □

Corollary 6.2.4. $\mathbb{D}R_{\mathfrak{R}} := \mathbb{D}_{\mathfrak{R}} \otimes \mathbb{C}[W]$ *is an associative algebra when equipped with the product*

$$(F \otimes w)(F' \otimes w') = F(w \cdot F') \otimes ww'$$

Proof. Let $F, F', F'' \in \mathbb{D}R_{\mathfrak{R}}$ and $w, w', w'' \in W$. Then

$$\begin{aligned}(F(w \cdot F'))(ww' \cdot F'') \otimes (ww')w'' &= F((w \cdot F')(ww' \cdot F'')) \otimes w(w'w'') \\ &= F(w \cdot (F'(w' \cdot F''))) \otimes w(w'w'') \\ &= (F \otimes w)(F'(w' \cdot F'') \otimes w'w'') \\ &= (F \otimes w)((F' \otimes w')(F'' \otimes w'')). \end{aligned}$$ □

The algebra $\mathbb{D}R_{\mathfrak{R}}$ is the algebra of $\mathfrak{R}$-valued differential reflection operators. It acts on $\mathbb{C}(H)$ by multiplication ($\mathfrak{R}$), derivation ($S(\mathfrak{h})$) and reflection ($\mathbb{C}[W]$), and similarly on all localisations of $\mathbb{C}[H]$ that contain $\mathfrak{R}$, in particular on $\mathbb{C}[H^{reg}]$. Furthermore, it also acts on the ring $\mathcal{O}(U)$ of holomorphic functions on a W-invariant open subset $U \subseteq H^{reg}$.

Proposition 6.2.5. *Let $P \in \mathbb{D}R_{\mathfrak{R}}$ such that $Pf = 0$ for all $f \in \mathbb{C}[H]$, then $P = 0$.*

Proof. Similar to [HS94, proposition 1.3.7]. □

Later we will also be interested in how these differential reflection operators act in different representations of W.

Definition 6.2.6. *Let (π, V) be a representation of W, then define*

$$\beta_\pi : \mathbb{D}R_{\mathfrak{R}} \to \mathbb{D}_{\mathfrak{R}} \otimes \mathrm{End}(V), \qquad \sum_{w \in W} p_w w \mapsto p_w \pi(w).$$

Proposition 6.2.7. *Let π satisfy $\pi(ww') = \pi(w'w)$ for all $w, w' \in W$, let $P, Q \in \mathbb{D}R_{\mathfrak{R}}$ and let $wQw^{-1} = Q$ for all $w \in W$. Then*

$$\beta_\pi(PQ) = \beta_\pi(P)\beta_\pi(Q).$$

Proof. Write

$$P = \sum_{w \in W} p_w w, \qquad Q = \sum_{w \in W} q_w w,$$

then Q's invariance implies that

$$\begin{aligned} wQw^{-1} &= \sum_{w' \in W} w q_{w'} w' w^{-1} \\ &= \sum_{w' \in W} (w \cdot q_{w'}) w w' w^{-1} \\ &= \sum_{w' \in W} (w \cdot q_{w^{-1}w'w}) w' \\ &= \sum_{w' \in W} q_{w'} w', \end{aligned}$$

i.e. that $w \cdot q_{w'} = q_{ww'w^{-1}}$ for all w, w'. Then

$$\begin{aligned}
\beta_\pi(PQ) &= \sum_{w,w' \in W} \beta_\pi(p_w w q_{w'} w') \\
&= \sum_{w,w' \in W} p_w (w \cdot q_{w'}) \pi(ww') \\
&= \sum_{w,w' \in W} p_w q_{ww'w^{-1}} \pi(ww') \\
&= \sum_{w,w' \in W} p_w q_{w'} \pi(w'w) \\
&= \sum_{w,w' \in W} p_w q_{w'} \pi(w) \pi(w') \\
&= \sum_{w \in W} p_w \pi(w) \sum_{w' \in W} q_{w'} \pi(w') \\
&= \beta_\pi(P) \beta_\pi(Q).
\end{aligned}$$

□

6.3. Dunkl Operator

Now for the protagonist of this chapter:

Definition 6.3.1. *Let $k = (k_\alpha)_{\alpha \in R}$ satisfy $k_\alpha = k_{w(\alpha)}$ for all $w \in W$, then we call k a* multiplicity vector*.*

Let k be a multiplicity vector and $\xi \in \mathfrak{h}$. The Dunkl–Cherednik operator *(henceforth mostly referred to as* Dunkl operator*) associated to this data is*

$$T_\xi(k) := \partial_\xi + \sum_{\alpha \in R^+} k_\alpha \alpha(\xi) \frac{1}{1 - e(-\alpha)} (1 - r_\alpha) - \rho(k)(\xi) \in \mathbb{D}R_{\mathfrak{R}}$$

for $2\rho(k) = \sum_{\alpha \in R^+} k_\alpha \alpha$.

Note that the two main sources for this chapter are [Opd00] and [HS94] who use different definitions of $T_\xi(k)$ that have very different properties but ultimately have sums of squares that reduce to the same differential operator in the trivial representation. We will be using [Opd00]'s definition and will only refer to [HS94]'s definition later on.

Proposition 6.3.2. *$T_\xi(k)$ acts on $\mathbb{C}[H]$.*

Proof. We have $\partial_\xi e(\lambda) = \lambda(\xi) e(\lambda)$, so the derivative operator leaves $\mathbb{C}[H]$ invariant. Same goes for the scalar $\rho(k)(\xi)$. So it remains to see that terms of the shape $\Delta_\alpha := (1 - e(-\alpha))^{-1}(1 - r_\alpha)$ leave $\mathbb{C}[H]$ invariant as well. Let $\lambda \in P$, we will show that $e(\lambda)$ is mapped to $\mathbb{C}[H]$.
"$\lambda(\alpha^\vee) = 0$": In this case we have $r_\alpha \cdot e(\lambda) = e(\lambda)$, so $\Delta_\alpha e(\lambda) = 0$.
"$n := \lambda(\alpha^\vee) > 0$": Then we have

$$(1 - r_\alpha) \cdot e(\lambda) = e(\lambda) - e(\lambda - n\alpha) = e(\lambda)(1 - e(-\alpha)^n).$$

This is evidently divisible by $1 - e(-\alpha)$ and we get

$$\Delta_\alpha e(\lambda) = e(\lambda)(1 + e(-\alpha) + \cdots + e((1-n)\alpha)).$$

"$-n := \lambda(\alpha^\vee) < 0$": Then we have

$$\begin{aligned}(1 - r_\alpha) \cdot e(\lambda) &= e(\lambda) - e(\lambda + n\alpha) = e(r_\alpha(\lambda))(e(-n\alpha) - 1) \\ &= -e(r_\alpha(\lambda))(1 - e(-n\alpha)).\end{aligned}$$

Similarly to before this is divisible by $1 - e(-\alpha)$ and yields

$$\Delta_\alpha e(\lambda) = -e(r_\alpha(\lambda))(1 + e(-\alpha) + \cdots + e((1-n)\alpha)).$$ □

Theorem 6.3.3. *Let k be a multiplicity vector, let $\xi, \eta \in \mathfrak{h}$, then $T_\xi(k)$ and $T_\eta(k)$ commute.*

Proof. This is the topic of [Opd00, section 2.3]. The idea of the proof is as follows: we know that $T_\xi(k), T_\eta(k)$ commute if they commute as endomorphisms of $\mathbb{C}[H]$. So for $k_\alpha \geq 0$ ($\alpha \in R$) we construct a basis of $\mathbb{C}[H]$ that simultaneously diagonalises all Dunkl operators (with the same multiplicity vector). This is done by constructing a monomial order and an inner product on $\mathbb{C}[H]$, with respect to which all $T_\xi(k)$ are upper triangular and symmetric. If we then take an ordered set of monomials, apply Gram–Schmidt orthogonalisation, we obtain a basis that diagonalises all $T_\xi(k)$ simultaneously. This implies that as endomorphisms of $\mathbb{C}[H]$, the Dunkl operators commute.

For more general multiplicity vectors, we note that the Dunkl operators depend algebraically on k, hence so does their commutator. This shows that the set of multiplicity vectors for which $[T_\xi(k), T_\eta(k)]$ for all $\xi, \eta \in \mathfrak{h}^*$ is Zariski-closed and contains the set of nonnegative multiplicity vectors. Since that set is already Zariski-dense, we conclude that our claim holds for all multiplicity vectors. □

6.4. Degenerate Affine Hecke Algebras

To each root system R we can associate a family of finitely generated algebras parametrised by multiplicity vectors.

Recall that we chose $\alpha_1, \dots, \alpha_n$ to be the simple roots of R^0. Define

$$k_i := \frac{1}{2} k_{\frac{1}{2}\alpha_i} + k_{\alpha_i}$$

where $k_{\frac{1}{2}\alpha_i}$ is zero if $\frac{1}{2}\alpha_i \notin R$.

Definition 6.4.1. *Let k be a multiplicity vector. The* degenerate affine Hecke algebra *(dAHA) associated to the root system R (and a subset R^+ of positive roots) and the multiplicity vector k is the associative algebra $H(R^+, k) := S(\mathfrak{h}) \otimes \mathbb{C}[W]$ with a multiplication such that*

$$S(\mathfrak{h}) \to H(R^+, k), p \mapsto p \otimes 1$$
$$\mathbb{C}[W] \to H(R^+, k), f \mapsto 1 \otimes f$$

are algebra homomorphisms, such that $pf = p \otimes f$ *for* $p \in S(\mathfrak{h}), f \in \mathbb{C}[W]$*, and such that*

$$r_i p = r_i(p) r_i - k_i \frac{p - r_i(p)}{\alpha_i^\vee}. \tag{6.1}$$

Proposition 6.4.2. *Let* k *be a multiplicity vector. The maps*

$$W \ni w \mapsto w, \qquad \mathfrak{h} \ni \xi \mapsto T_\xi(k)$$

extend to a ring homomorphism $H(R^+, k) \to \mathbb{D}R_{\mathfrak{R}}$ *that maps to the subring of operators that leave* $\mathbb{C}[H]$ *invariant.*

Proof. Evidently, the map $W \to \mathbb{D}R_{\mathfrak{R}}$ extends linearly to $\mathbb{C}[W]$. Because of Theorem 6.3.3, the map $\mathfrak{h} \to \mathbb{D}R_{\mathfrak{R}}$ extends to $S(\mathfrak{h})$. Thus, it remains to show that the commutation relations between $\mathbb{C}[W]$ and $S(\mathfrak{h})$ match. Let $i = 1, \ldots, n$ and let $p \in S(\mathfrak{h})$ be homogeneous. Let's show the relation (6.1) for r_i and p by induction in the degree of p.
"deg(p) = 0": In this case $r_i(p) = p$, so that in the Hecke algebra we have $r_i p = p r_i$. Similarly, in terms of differential reflection operators, p just maps to a constant, which commutes with $\mathbb{C}[W]$.
"deg(p) = 1": Let $p = \xi \in \mathfrak{h}$. Then

$$\begin{aligned}
r_i T_\xi(k) r_i &= r_i \partial_\xi r_i + \sum_{\alpha \in R^+} k_\alpha \alpha(\xi) r_i \Big(\frac{1}{1 - e(-\alpha)}(1 - r_\alpha) - \frac{1}{2} \Big) r_i \\
&= \partial_{r_i(\xi)} + \sum_{\alpha \in R^+} k_\alpha \alpha(\xi) \Big(\frac{1}{1 - e(-r_i(\alpha))}(1 - r_i r_\alpha r_i) - \frac{1}{2} \Big) \\
&= \partial_{r_i(\xi)} + \sum_{\alpha \in R^+} k_\alpha \alpha(\xi) \Big(\frac{1}{1 - e(-r_i(\alpha))}(1 - r_{r_i(\alpha)}) - \frac{1}{2} \Big) \\
&= \partial_{r_i(\xi)} + \sum_{\alpha \in R^+} k_{r_i(\alpha)} r_i(\alpha)(r_i(\xi)) \Big(\frac{1}{1 - e(-r_i(\alpha))}(1 - r_{r_i(\alpha)}) - \frac{1}{2} \Big)
\end{aligned}$$

the latter due to k's Weyl invariance, and because of how W acts on $\mathfrak{h}^*$ and $\mathfrak{h}$. Substituting $r_i(\alpha) \to \alpha$ yields

$$\partial_{r_i(\xi)} + \sum_{\alpha \in r_i(R^+)} k_\alpha \alpha(r_i(\xi)) \Big(\frac{1}{1 - e(-\alpha)}(1 - r_\alpha) - \frac{1}{2} \Big).$$

By Lemma 6.1.7, we know what $r_i(R^+)$ looks like. In particular, since we took $k_{\frac{1}{2}\alpha_i} = 0$ if $\frac{1}{2}\alpha_i \notin R$, we don't particularly need to know if α_i is divisible or not. In both cases we get

$$\begin{aligned}
r_i T_\xi(k) r_i &= T_{r_i(\xi)}(k) - k_{\alpha_i}\alpha_i(r_i(\xi))\left(\frac{1}{1-e(-\alpha_i)}(1-r_i) - \frac{1}{2}\right) \\
&\quad - k_{\frac{1}{2}\alpha_i}\frac{1}{2}\alpha_i(r_i(\xi))\left(\frac{1}{1-e\left(-\frac{1}{2}\alpha_i\right)}(1-r_i) - \frac{1}{2}\right) \\
&\quad + k_{\alpha_i}(-\alpha_i)(r_i(\xi))\left(\frac{1}{1-e(\alpha_i)}(1-r_i) - \frac{1}{2}\right) \\
&\quad + k_{\frac{1}{2}\alpha_i}\left(-\frac{1}{2}\alpha_i\right)(r_i(\xi))\left(\frac{1}{1-e\left(\frac{1}{2}\alpha_i\right)} - \frac{1}{2}\right) \\
&= T_{r_i(\xi)}(k) + \alpha_i(\xi)\left(-k_{\alpha_i} - \frac{1}{2}k_{\frac{1}{2}\alpha_i} + k_{\alpha_i}(1-r_i) + \frac{1}{2}k_{\frac{1}{2}\alpha_i}(1-r_i)\right) \\
&= T_{r_i(\xi)}(k) - k_i\alpha_i(\xi) r_i \\
&= T_{r_i(\xi)}(k) - k_i\frac{\xi - (\xi - \alpha_i(\xi)\alpha_i^\vee)}{\alpha_i^\vee} r_i.
\end{aligned}$$

Multiplying with r_i on the right gives

$$r_i T_\xi(k) = T_{r_i(\xi)}(k) r_i - k_i \frac{\xi - r_i(\xi)}{\alpha_i^\vee}$$

as desired.

"$\deg(p) \to \deg(p{+}1)$": Let $\xi \in \mathfrak{h}$. Write $p(T(k))$ for the polynomial of Dunkl operators obtained for p. Then by induction hypothesis (using the $\deg(p)$ and 1 cases)

$$\begin{aligned}
r_i T_\xi(k) p(T(k)) &= T_{r_i(\xi)}(k) r_i p(T(k)) - k_i \frac{\xi - r_i(\xi)}{\alpha_i^\vee}(T(k)) p(T(k)) \\
&= T_{r_i(\xi)}(k) r_i(p)(T(k)) r_i - k_i T_{r_i(\xi)}(k)\frac{p - r_i(p)}{\alpha_i^\vee}(T(k)) \\
&\quad - k_i \frac{\xi - r_i(\xi)}{\alpha_i^\vee}(T(k)) p(T(k)) \\
&= r_i(\xi p)(T(k)) r_i - k_i \frac{\xi p - r_i(\xi p)}{\alpha_i^\vee}(T(k)). \qquad \square
\end{aligned}$$

In particular, the Hecke algebra exists, is an associative algebra, and can be realised as subalgebra of $\mathrm{End}(\mathbb{C}[H])$.

Corollary 6.4.3. *In particular, any $\mathbb{D}R_{\mathfrak{R}}$-module is a $H(R^+, k)$-module. This includes, $\mathbb{C}[H], \mathbb{C}[H^{reg}], \mathbb{C}[H]$, as well as $\mathcal{O}(U)$ for $U \subseteq H^{reg}$ open and W-invariant.*

6.5. Representation Theory of dAHAs

In order to understand $H(R^+, k)$-modules better, our first goal is to describe what $H(R^+, k)$'s central characters look like.

6.5.1. Central Characters

Proposition 6.5.1. *The centre of* $H(R^+, k)$ *is* $S(\mathfrak{h})^W$.

Proof. Let $P \in Z(H(R^+, k))$, say

$$P = \sum_{w \in W} p_w w.$$

Let $\xi \in \mathfrak{h}$ be regular, then

$$P\xi - \xi P = \sum_{w \in W} p_w(w\xi - \xi w) = \sum_{w \in W} p_w(w(\xi) - \xi)w + \text{lower-order terms}.$$

The highest-order term in front of w will be $p_w(w(\xi) - \xi)$, which has to be zero. For $w \neq 1$, this implies $p_w = 0$, so that $P \in S(\mathfrak{h})$.

Next up, we know that $P = p_1$ has to commute with all r_i, so that

$$r_i p_1 - p_1 r_i = (r_i(p_1) - p_1)r_i - \frac{k_i}{\alpha_i^\vee}(p_1 - r_i(p_1)) = 0.$$

Since $r_i, 1$ are linearly independent over $S(\mathfrak{h})$, we obtain $r_i(p_1) = p_1$, hence that p_1 is invariant under the group generated by all the r_i, i.e. by W. □

Proposition 6.5.2. *(a) Every algebra morphism* $S(\mathfrak{h})^W$ *is given by*

$$\chi_\lambda : z \mapsto z(\lambda)$$

for some $\lambda \in \mathfrak{h}^*$.

(b) Where $\chi_\lambda = \chi_\mu$ *(on* $S(\mathfrak{h})^W$*) iff* $W\lambda = W\mu$.

Proof. (a) Let $\chi : S(\mathfrak{h})^W \to \mathbb{C}$ be an algebra homomorphism. If $\chi = 0$, take $\lambda = 0$. Otherwise, χ is surjective and we have $S(\mathfrak{h})^W / \ker(\chi) \cong \mathbb{C}$. This shows that $\ker(\chi)$ is an ideal whose ring of residues is a field. Thus, $\ker(\chi)$ is a maximal ideal.

Write now $s : S(\mathfrak{h}) \to S(\mathfrak{h})^W$ for the symmetriser (linear map)

$$p \mapsto \frac{1}{\#W} \sum_{w \in W} w(p).$$

For $p \in S(\mathfrak{h}), z \in S(\mathfrak{h})^W$ we have $s(pz) = s(p)z = s(p)s(z)$.

Let $p_1, \ldots, p_r \in S(\mathfrak{h}), z_1, \ldots, z_r \in S(\mathfrak{h})^W$ such that $\sum_{i=1}^r p_i z_i = 1$, then

$$1 = \chi(s(1)) = \chi\left(s\left(\sum_{i=1}^r s_i z_i\right)\right) = \sum_{i=1}^r \chi(s(s_i z_i)) = \sum_{i=1}^r \chi(s(s_i)z_i) = 0,$$

so there is no $S(\mathfrak{h})$-linear combination of elements of $\ker(\chi)$ that is 1. This shows that $I := S(\mathfrak{h}) \ker(\chi)$ is a proper ideal of $S(\mathfrak{h})$. As a consequence of Zorn's lemma, there is a maximal ideal $\mathfrak{m}$ containing I. Then, $\Updownarrow \cap S(\mathfrak{h})^W$ is a proper ideal of $S(\mathfrak{h})^W$ that contains $\ker(\chi)$, hence it equals $\ker(\chi)$.

By Hilbert's Nullstellensatz, there is now a $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{m} = \ker(\chi_\lambda)$. In particular, for every $z \in S(\mathfrak{h})^W$ we have

$$\begin{aligned}\chi(z)1 + \mathfrak{m} &= (\chi(z)1 + (\mathfrak{m} \cap S(\mathfrak{h})^W)) + \mathfrak{m} \\ &= (\chi(z)1 + \ker(\chi)) + \mathfrak{m} \\ &= (z + \ker(\chi)) + \mathfrak{m} \\ &= z + \mathfrak{m} \\ &= \chi_\lambda(z)1 + \mathfrak{m},\end{aligned}$$

hence $\chi = \chi_\lambda|_{S(\mathfrak{h})^W}$.

(b) "$\Leftarrow$": Evidently χ_λ and $\chi_{w(\lambda)}$ are equal on $S(\mathfrak{h})^W$ for all $w \in W$.
"$\Rightarrow$": Assume now that χ_λ, χ_μ are equal on $S(\mathfrak{h})^W$. Let $\mathfrak{m}, \mathfrak{m}'$ be the kernels (in $S(\mathfrak{h})$) of χ_λ, χ_μ, respectively. Since χ_λ and χ_μ are equal on invariants, we have $\mathfrak{m} \cap S(\mathfrak{h})^W = \mathfrak{m}' \cap S(\mathfrak{h})^W$.

Let

$$f_1 \in \bigcap_{w \in W} w(\mathfrak{m}), \qquad f_2 \in \bigcap_{w \in W} w(\mathfrak{m}'),$$

then $s(f_1) \in \mathfrak{m}$ and $s(f_2) \in \mathfrak{m}'$ are both invariant, hence

$$s(f_1 + f_2) = s(f_1) + s(f_2) \in \mathfrak{m} \cap S(\mathfrak{h})^W = \mathfrak{m}' \cap S(\mathfrak{h})^W \neq S(\mathfrak{h})^W.$$

In particular, $s(f_1 + f_2) \neq 1$, which implies that $f_1 + f_2 \neq 1$ (since 1 is invariant). Thus,

$$1 \notin \bigcap_{w \in W} w(\mathfrak{m}) + \bigcap_{w \in W} w(\mathfrak{m}') = \bigcap_{w, w' \in W} (w(\mathfrak{m}) + w'(\mathfrak{m}')).$$

This shows that there are $w, w' \in W$ such that $w(\mathfrak{m}) + w'(\mathfrak{m}') \neq (1)$, and since they are both maximal ideals that means that they are equal. This then implies $w(\lambda) = w'(\mu)$, and hence that λ, μ lie in the same W-orbit.

□

6.5.2. Intertwiners

Our goal is to understand some/most of the simple $H(R^+, k)$-modules and one way of achieving this is to look how some subalgebras of $H(R^+, k)$ act on them, in particular, by analysing them as $\mathbb{C}[W]$-modules and as $S(\mathfrak{h})$-modules. Let M be a $H(R^+, k)$-module, let χ be a central character of $\mathbb{C}[W]$, then write $M[\chi]$ for the isotypic component as $\mathbb{C}[W]$-modules; in particular write M^W for the isotypic component associated with the trivial representation.

For $\lambda \in \mathfrak{h}^*$ write

$$M^\lambda := \{v \in M \mid \forall p \in S(\mathfrak{h}) : p \cdot v = p(\lambda)v\}$$

for the isotypic component with central character χ_λ of the $S(\mathfrak{h})$-module M. M^λ is called the λ-*weight space*.

We'd like to establish some relations among the weight spaces and between the weight spaces and M^W. For that it is useful to look for elements of $H(R^+, k)$ that intertwine between the weight spaces, i.e. $P \in H(R^+, k)$ such that $PM^\lambda = M^{w(\lambda)}$ for some $w \in W$.

Definition 6.5.3. *Let $i = 1, \ldots, n$, then define*

$$I_i := r_i \alpha_i^\vee + k_i = -\alpha_i^\vee r_i - k_i.$$

Recall that $\alpha_1, \ldots, \alpha_n$ are the simple roots of R^0 and

$$k_i = \frac{1}{2} k_{\frac{1}{2}\alpha_i} + k_{\alpha_i}.$$

These $I_1, \ldots, I_n$ are going to be the intertwiners associated to the simple reflections $r_1, \ldots, r_n$. To see that we can define intertwiners for every element of W, we need to establish some properties first. For that, let's begin with a few lemmas that are going to be useful.

Lemma 6.5.4. *Define*

$$F_i := \operatorname{span}\{pw \mid w \in W, p \in S(\mathfrak{h}), \ell(w) \le i\}.$$

Then $(F_i)_{i \in \mathbb{N}_0}$ is a filtration on $H(R^+, k)$.

Here $\ell(w)$ refers to the shortest possible length *of an expression of w in terms of the r_i. This means that*

$$\ell(w) = \min\{r \mid \exists i_1, \ldots, i_r \in \{1, \ldots, n\} : w = r_{i_1} \cdots r_{i_r}\}.$$

Any expression $r_{i_1} \cdots r_{\ell(w)} = w$ (of length $\ell(w)$) is called reduced.

Proof. Evidently we have $F_i \subseteq F_{i+1}$ and $F_{\#W} = H(R^+, k)$ because

$$\{pw \mid w \in W, p \in S(\mathfrak{h}), \ell(w) \le \#W\}$$

contains all elementary tensors of $S(\mathfrak{h}) \otimes \mathbb{C}[W]$.

So we just need to show that for $pw \in F_r$ and $p'w' \in F_s$ we have $pwp'w' \in F_{r+s}$. We do this by induction on r.
"$r = 0$": We have $pw = p$, then for any s, $pwp'w' = pp'w' \in F_s = F_{r+s}$.
"$n \to n+1$": Let $i = 1, \ldots, n$, $pw \in F_r$, and $p'w' \in F_s$ for any $s \in \mathbb{N}_0$, then

$$pwr_i p'w' = pw(r_i \cdot p')r_i w' - k_i pw \frac{p' - r_i(p')}{\alpha_i^\vee} w'.$$

Since $\ell(r_i w') \le \ell(r_i) + \ell(w') = s + 1$, we can write those two terms as the products of

$$pw, (r_i \cdot p')r_i w' \qquad \text{and} - k_i pw, \frac{p' - r_i(p')}{\alpha_i^\vee} w'$$

respectively. In both cases, the first term lies in F_r (the second has degrees $\le s+1$ and $\le s$, respectively), so we can apply the induction hypothesis. $\square$

Corollary 6.5.5. *Let* $\mathrm{gr}(F_\bullet)$ *be the graded algebra associated to the filtration* $(F_i)_{i\in\mathbb{N}_0}$ *of* $H(R^+, k)$*. For* $w \in W, p \in S(\mathfrak{h})$ *we have*

$$wp = w(p)w$$

in $\mathrm{gr}(F_\bullet)$*.*

Proof. By induction in $\ell(w)$.
"$\ell(w) = 0$": We have $1p = p1$.
"$\ell(w) \to \ell(w)+1$": Let $i = 1, \ldots, n$ such that $\ell(wr_i) = \ell(w) + 1 =: r + 1$. Then we have

$$\begin{aligned} wr_ip + F_r &= w\left(r_i(p)r_i - k_i\frac{p - r_i(p)}{\alpha_i^\vee}\right) + F_r \\ &= wr_i(p)r_i - k_iw\frac{p - r_i(p)}{\alpha_i^\vee} + F_r \\ &= wr_i(p)r_i + F_r \\ &= w(r_i(p))wr_i + F_r \end{aligned}$$

by the induction hypothesis. □

Lemma 6.5.6. *Let* $r_{i_1} \cdots r_{i_r} = w$ *be a reduced expression, then*

$$I_{i_1} \cdots I_{i_r} + F_{r-1} = \left(\prod_{\alpha \in R^{0,-} \cap w(R^{0,+})} \alpha^\vee\right) w + F_{r-1}.$$

Proof. By induction in r.
"$r = 0$": the identity maps no roots from $R^{0,+}$ to $R^{0,-}$, hence we have 1 on both sides.
"$r \to r+1$": Let $\ell(r_iw) = \ell(w)+1$, then by induction hypothesis and using Corollary 6.5.5 we have

$$\begin{aligned} I_iI_{i_1} \cdots I_{i_r} + F_r &= (-\alpha_i^\vee r_i - k_i)\left(\prod_{\alpha \in R^{0,-} \cap w(R^{0,+})} \alpha^\vee\right) w + F_r \\ &= (-\alpha_i^\vee)r_i\left(\prod_{\alpha \in R^{0,-} \cap w(R^{0,+})} \alpha^\vee\right) w + F_r \\ &= (-\alpha_i^\vee)\left(\prod_{\alpha \in R^{0,-} \cap w(R^{0,+})} r_i(\alpha)^\vee\right) r_iw + F_r \\ &= (-\alpha_i^\vee)\left(\prod_{\alpha \in r_i(R^{0,-} \cap w(R^{0,+}))} \alpha^\vee\right) r_iw + F_r. \end{aligned}$$

Since r_i is bijective, we have $r_i(R^{0,-} \cap w(R^{0,+})) = r_i(R^{0,-}) \cap r_i(w(R^{0,+}))$. By Lemma 6.1.7, we have $r_i(R^{0,-}) = R^{0,-} \setminus \{-\alpha_i\} \cup \{\alpha_i\}$. Furthermore, since $\ell(r_iw) > \ell(w)$, we have

$\alpha_i \in w(R^{0,+})$ and hence $-\alpha_i \in r_i(w(R^{0,+}))$. This shows that

$$\begin{aligned} r_i(R^{0,-} \cap w(R^{0,+})) &= \left(R^{0,-} \setminus \{-\alpha_i\} \cup \{\alpha_i\}\right) \cap r_i(w(R^{0,+})) \\ &= R^{0,-} \cap r_i(w(R^{0,+})) \setminus \{-\alpha_i\}\,, \end{aligned}$$

and hence that

$$I_i I_{i_1} \cdots I_{i_r} + F_r = \left(\prod_{\alpha \in R^{0,-} \cap r_i(w(R^{0,+}))} \alpha^\vee \right) r_i w + F_r. \qquad \square$$

Theorem 6.5.7. *The intertwiners have the following properties:*

(a) $I_i^2 = k_i^2 - (\alpha_i^\vee)^2$.

(b) For $p \in S(\mathfrak{h})$ *we have* $I_i p = r_i(p) I_i$.

(c) Let $i \neq j$ *and* m_{ij} *be the order of* $r_i r_j \in W$. *Then*

$$I_i I_j \cdots = I_j I_i \ldots \qquad (m_{ij}\text{-many terms}),$$

i.e. $I_1, \ldots, I_n$ *satisfy the same braid relations as* $r_1, \ldots, r_n$.

Proof. (a)

$$\begin{aligned} I_i^2 &= (r_i \alpha_i^\vee + k_i)^2 = r_i \alpha_i^\vee r_i \alpha_i^\vee + 2 r_i \alpha_i^\vee k_i + k_i^2 \\ &= (-\alpha_i^\vee r_i - k_i \alpha_i(\alpha_i^\vee)) r_i \alpha_i^\vee + 2 r_i \alpha_i^\vee k_i + k_i^2 \\ &= k_i^2 - (\alpha_i^\vee)^2 \end{aligned}$$

because $\alpha_i(\alpha_i^\vee) = 2$.

(b) Similarly to the proof of Proposition 6.4.2, we can show this by induction and reduce the problem to the homogeneous degree 1 case. Let $\xi \in \mathfrak{h}$, then

$$\begin{aligned} I_i \xi &= (r_i \alpha_i^\vee + k_i)\xi = r_i \xi \alpha_i^\vee + \xi k_i \\ &= (r_i(\xi) r_i - k_i \alpha_i(\xi))\alpha_i^\vee + \xi k_i \\ &= r_i(\xi) r_i \alpha_i^\vee + (\xi - \alpha_i(\xi)\alpha_i^\vee) k_i \\ &= r_i(\xi)(r_i \alpha_i^\vee + k_i) \\ &= r_i(\xi) I_i. \end{aligned}$$

(c) The statement is equivalent to

$$I_{i_1} \cdots I_{i_r} = I_{j_1} \cdots I_{j_r}$$

if

$$r_{i_1} \cdots r_{i_r} = r_{j_1} \cdots r_{j_r} = w$$

are two equal reduced expressions. Let I, J be the products of intertwiners on the left and on the right, respectively. From Lemma 6.5.6 we know their leading term with respect to $F_\bullet$:

$$I + F_{r-1} = J + F_{r-1} = \left(\prod_{\alpha \in R^{0,-} \cap w(R^{0,+})} \alpha^\vee \right) w + F_{r-1}.$$

Thus, $I - J \in F_{r-1}$, say

$$I - J = \sum_{\ell(w') < \ell(w)} p_{w'} w' \qquad (p_{w'} \in S(\mathfrak{h})).$$

We are now showing that $I - J$ is contained in all subspaces F_i of the filtration $F_\bullet$ and hence is zero. For $i = r - 1$, we already know that.
"$i \to i - 1$": By induction hypothesis we have

$$I - J + F_{i-1} = \sum_{\ell(w') = i} p_{w'} w'.$$

Let $\xi \in \mathfrak{h}$ be regular, then by Corollary 6.5.5 we have

$$(I - J)\xi + F_{i-1} = \sum_{\ell(w') = i} p_{w'} w' \xi + F_{i-1} = \sum_{\ell(w') = i} p_{w'} w'(\xi) w' + F_{i-1}.$$

By assumption this is now also equal to

$$w(\xi)(I - J) = \sum_{\ell(w') = i} p_{w'} w(\xi) w' + F_{i-i}.$$

Since the w' are linearly independent, this shows that $p_{w'} w(\xi) = p_{w'} w'(\xi)$ for every w' of length i. Since ξ is regular, we have $w(\xi) \neq w'(\xi)$, and thus $p_{w'} = 0$.

□

With these properties out of the way, we can make the following definition:

Definition 6.5.8. *Let $w = r_{i_1} \cdots r_{i_r} \in W$ be a reduced expression. Define*

$$I_w := I_{i_1} \cdots I_{i_r},$$

this is the intertwiner *for w.*

By Theorem 6.5.7(c), this definition is unambiguous. It follows directly that $I_{ww'} = I_w I_{w'}$ if $\ell(ww') = \ell(w) + \ell(w')$. Otherwise, this is not necessarily true, as we can see for example from $I_i^2 \neq 1$ (Theorem 6.5.7(a)). Nevertheless, we still have $I_w p = w(p) I_w$ for all $p \in S(\mathfrak{h})$, which is what we set out to do.

6.5.3. Induced Simple Modules

We are now going to construct something similar to the induced representations from Section 4, only purely algebraic, and put our theory of intertwiners to good use.

Definition 6.5.9. *Let $\lambda \in \mathfrak{h}^*$, write $\mathbb{C}_\lambda$ for the 1-dimensional $S(\mathfrak{h})$-module with $p \cdot 1 = p(\lambda)$. Then define*

$$\mathcal{I}_\lambda := H(R^+, k) \otimes_{S(\mathfrak{h})} \mathbb{C}_\lambda.$$

$\mathcal{I}_\lambda$ is evidently a $H(R^+, k)$-module with central character χ_λ. Furthermore, as $\mathbb{C}[W]$-modules we have $\mathcal{I}_\lambda \cong \mathbb{C}[W]$, so $\mathcal{I}$ is in particular $\#W$-dimensional.

Lemma 6.5.10. *Let λ be regular and satisfy $\mu(\alpha_i^\vee) \neq \pm k_i$ ($\mu \in W\lambda$). Then*

(a) As an $S(\mathfrak{h})$-module it decomposes as

$$\mathcal{I}_\lambda = \bigoplus_{\mu \in W\lambda} \mathcal{I}_\lambda^\mu,$$

(b) $\mathcal{I}_\lambda$ is simple.

(c) Every finite-dimensional simple $H(R^+, k)$-module with central character χ_λ is isomorphic to $\mathcal{I}_\lambda$.

(d) The map $\mathcal{I}_\lambda^\mu \to \mathcal{I}_\lambda^W$ given by the action of $\sum_{w \in W} w$ is a linear isomorphism.

Proof. (a) Note that for $\mu \neq \mu' \in W\lambda$ we can find a vector ξ with $\mu(\xi) \neq \mu'(\xi)$. For every vector $v \in \mathcal{I}_\lambda^\mu \cap \mathcal{I}_\lambda^{\mu'}$ we therefore have

$$\mu(\xi)v = \xi \cdot v = \mu'(\xi)v,$$

which establishes that $v = 0$, thus the sum is direct.

To show that the sum is indeed all of $\mathcal{I}_\lambda$, we proceed by a dimension argument. Since $1 \in \mathcal{I}_\lambda^\lambda$, we have $\dim(\mathcal{I}_\lambda^\lambda) \geq 1$. By Theorem 6.5.7(b), $I_w \mathcal{I}_\lambda^\mu \subseteq \mathcal{I}_\lambda^{w(\mu)}$ for any $w \in W, \mu \in W\lambda$, so the intertwiners are $S(\mathfrak{h})$-morphisms between the weight spaces of $\mathcal{I}_\lambda$.

Note that on $\mathcal{I}_\lambda^\mu$ we have $I_i^2 = k_i^2 - \mu(\alpha_i^\vee)^2$, which is nonzero (hence invertible) by assumption. As a consequence, all I_i ($i = 1, \dots, n$) and all I_w ($w \in W$) are bijections, hence isomorphisms of $S(\mathfrak{h})$-modules. As a consequence, all weight spaces $\mathcal{I}_\lambda^\mu$ are in bijection to $\mathcal{I}_\lambda^\lambda$, hence all have dimension ≥ 1. Therefore, the direct sum of all weight spaces has dimension $\geq \#W$, so it equals all of $\mathcal{I}_\lambda$.

(b) Let $M \leq \mathcal{I}_\lambda$ be a nontrivial submodule, then the action of $S(\mathfrak{h})$ on M gives rise to a generalised weight space decomposition, with at least one weight vector v of some weight μ. Since $\mathcal{I}$ has central character χ_λ, we have $\chi_\mu = \chi_\lambda$ on $S(\mathfrak{h})^W$, which by Proposition 6.5.2(b) implies that $\mu \in W\lambda$. Thus, by the intertwiner action we get $0 \neq I_w v \in M^{w(\mu)}$, so $\dim(M) \geq \#W = \dim(\mathcal{I}_\lambda)$. Therefore, $M = \mathcal{I}_\lambda$.

(c) Let M be a simple $H(R^+, k)$-module with central character χ_λ and dimension $\leq \#W$. The action of $S(\langle)$ induces a generalised weight space decomposition

$$\bigoplus_\mu \sum_{i=1}^{\infty} M^{\lambda,n}$$

where

$$M^{\lambda,n} = \{v \in M \mid \forall p \in S(\mathfrak{h}) : (p - p(\lambda))^n \cdot v = 0\}.$$

In particular, there is at least one weight vector v for some weight μ. By a similar argument as for the simplicity proof, $\mu \in W\lambda$, and we can define a $H(R^+, k)$-linear map $\phi : \mathcal{I}_\lambda \to M$ by having $\phi(1) = I_w v$ where $w(\mu) = \lambda$. This is well-defined because

$$\begin{aligned} p\phi(1) = pI_w v = I_w w^{-1}(p)v = I_w w^{-1}(p)(\mu)v \\ = w^{-1}(p)(w^{-1}(\lambda))I_w v = p(\lambda)\phi(1) = \phi(p \cdot 1). \end{aligned}$$

By Schur's Lemma 2.3.6(a), this is now an isomorphism.

(d) Let $0 \neq v \in \mathcal{I}_\lambda^\mu$. Since $\mathcal{I}_\lambda$ is simple, $H(R^+, k)v = \mathcal{I}_\lambda$. Since $S(\mathfrak{h})$ acts by means of μ, this really means that $\mathbb{C}[W]v = \mathcal{I}_\lambda$. If $\sum_{w \in W} w \cdot v = 0$, this shows that the $w \cdot v$ ($w \in W$) are linearly dependent, so that $\mathbb{C}[W]v$ has dimension $< \#W = \dim(\mathcal{I}_\lambda)$, which is a contradiction. Thus, the symmetrisation map is injective. Since $\mathbb{C}[W] \cong \mathcal{I}_\lambda$ $\mathbb{C}[W]$-modules, we have $\dim(\mathcal{I}_\lambda^W) = 1$, hence the symmetrisation map is also surjective.

□

Lemma 6.5.11. *Let M be any $H(R^+, k)$-module with central character χ_λ, where we have $\mu(\alpha_i^\vee) \neq 0, \pm k_i$ for $i = 1, \ldots, n$ and all $\mu \in W\lambda$. The map*

$$M^\lambda \to M^W : v \mapsto \sum_{w \in W} w \cdot v,$$

i.e. action by $\sum_{w \in W} w$, is then a $\mathbb{C}$-linear isomorphism and there is an index set I such that

$$M \cong \mathcal{I}_\lambda^{(I)}.$$

Proof. By [Che55], there are $\#W$-many elements $1 = h_1, \ldots, h_{\#W}$ of $S(\mathfrak{h})$ that generate $S(\mathfrak{h})$ as a $S(\mathfrak{h})^W$-module. If $q \in M^W$, this therefore implies that $H(R^+, k)q$ is $\leq \#W$-dimensional because the action of a general element of $H(R^+, k)$ on q looks like

$$\sum_{i=1}^{\#W} \sum_{w \in W} s_{i,w} h_i w q = \sum_{i=1}^{\#W} \sum_{w \in W} s_{i,w} h_i q = \sum_{i=1}^{\#W} \sum_{w \in W} s_{i,w}(\lambda) h_i q,$$

so that $h_1 q, \ldots, h_{\#W} q$ is a $\mathbb{C}$-generating system of that submodule. By an argument similar to Lemma 6.5.10(b), this shows that $H(R^+, k)q$ is simple, and therefore isomorphic to $\mathcal{I}_\lambda$.

If $(q_i)_{i\in I}$ is a basis of M^W, we then have

$$M \supseteq \bigoplus_{i\in I} H(R^+,k)q_i \cong \mathcal{I}_\lambda^{(I)}.$$

To see that this is indeed everything, let $v \in M$ and consider the cyclic module $H(R^+,k)v$. If $J \subseteq H(R^+,k)$ is the annihilator of v (left ideal), then $H(R^+,k)v \cong H(R^+,k)/J$. Since M has central character χ_λ, we know that the ideal

$$J_\lambda := \langle p - p(\lambda) | p \in S(\mathfrak{h})^W \rangle$$

is contained in J. In particular, $H(R^+,k)v$ is a quotient of $H(R^+,k)/J_\lambda$. A basis for $H(R^+,k)/J_\lambda$ can be given by $h_i \otimes w$ ($i = 1,\dots,\#W, w \in W$), so that $H(R^+,k)/J_\lambda$ is $(\#W)^2$-dimensional and has $\#W$-many invariant vectors. By a similar argument to before, we can find a direct sum of simple $H(R^+,k)$-modules inside $H(R^+,k)/J_\lambda$, but now, by a dimension argument we can actually establish equality: $H(R^+,k)/J_\lambda \cong \mathcal{I}_\lambda^{\#W}$. Thus, $H(R^+,k)v$ is a cyclic quotient module of a semisimple module, hence simple. This means that $H(R^+,k)v$ is generated by a W-invariant element, hence it is a submodule of

$$\bigoplus_{i\in I} H(R^+,k)q_i.$$

This shows that M is semisimple and isotypic. By Lemma 6.5.10(d), the symmetrisation map is a linear isomorphism. □

Corollary 6.5.12. *Let M be a $H(R^+,k)$-module with central character χ_λ (for $\mu(\alpha_i^\vee) \neq 0, \pm k_i$ for all $\mu \in W\lambda$), then* $\dim(M) = \#W \dim(M^W)$ *(both sides potentially infinite).*

Proof. By the proof of Lemma 6.5.11, $M \cong \mathcal{I}_\lambda^{(I)}$ for $\#I = \dim(M^W)$. Thus, we have

$$\dim(M) = \dim(\mathcal{I}_\lambda^{(I)}) = \#I \dim(\mathcal{I}_\lambda) = \dim(M^W)\#W.$$ □

6.5.4. Hypergeometric Function

Let's now apply this abstract representation theory of $H(R^+,k)$ to some actual concrete examples of $H(R^+,k)$-modules. To begin with, we can use some of the machinery already developed in the proof of Theorem 6.3.3 and milk it a bit:

Theorem 6.5.13 (Polynomials)**.** *Recall that $H(R^+,k)$ acts on $\mathbb{C}[H]$. Then*

$$\mathbb{C}[H] = \bigoplus_{\lambda\in\rho(k)+P^+} \mathbb{C}[H][\chi_\lambda],$$

where $\mathbb{C}[H][\chi_\lambda] \cong \mathbb{C}[W/\operatorname{Stab}(\lambda)]$ as $\mathbb{C}[W]$-modules.

Sketch of Proof. See [Opd00, propositions 5.4, 5.5]. We use the orthonormal basis from the proof of Theorem 6.3.3 and analytically continue them for arbitrary multiplicity vectors k. They simultaneously diagonalise all Dunkl operators. In particular the polynomial with leading term related to $\mu \in W\lambda$ ($\lambda \geq 0$) has a weight in $W(\lambda+\rho)$. Thus every basis element lies in a χ_λ-isotypic component for $\lambda \in \rho(k) + P^+$. □

In particular, for every $\lambda \in P^+$ there is exactly one W-invariant polynomial in $\mathbb{C}[H][\chi_{\rho(k)+\lambda}]$ with leading coefficient (in front of $e(\lambda)$) 1. This polynomial is called the *Jacobi polynomial* $P(\lambda, k)$.

As is somewhat to be expected, these polynomials only cover a discrete range of parameters. For the more general picture, we have to consider holomorphic functions on all possible domains. For that note that

Proposition 6.5.14. *The sheaf $H^{reg} \supseteq U \mapsto \mathcal{O}(U)$ of holomorphic functions on the site of W-invariant opens $U \subseteq H^{reg}$ is a sheaf of $\mathbb{D}R_{\mathfrak{R}}$-modules.*

Proof. Note that the W-invariant opens of H^{reg} form a site that is isomorphic to the little Zariski site (site of open subsets) of H^{reg}/W, and that $U \mapsto \mathcal{O}(U)$ is a sheaf (of $\mathbb{C}$-algebras) on that site because holomorphicity is a purely local condition, so we can glue holomorphic functions that agree on overlaps.

Let $U' \subseteq U$ be W-invariant open subsets of H^{reg}, let $f \in \mathcal{O}(U)$, then

$$\partial_p(f|_{U'}) = (\partial_p f)|_{U'}$$
$$\frac{1}{1-e(-\alpha)} f|_{U'} = \left.\frac{f}{1-e(-\alpha)}\right|_{U'}$$
$$r_i(f|_{U'}) = r_i(f)|_{U'},$$

so that the restriction map $\mathcal{O}(U) \to \mathcal{O}(U')$ is a morphism of $\mathbb{D}R_{\mathfrak{R}}$-modules. □

Corollary 6.5.15. *$\mathcal{O}$ is a sheaf of $H(R^+, k)$-modules for every multiplicity vector k.*

Definition 6.5.16. *For $U \subseteq H^{reg}$ W-invariant define*

$$\mathcal{S}(\lambda, k)(U) := \mathcal{O}(U)[\chi_\lambda]$$

(as $H(R^+, k)$-modules) the sheaf of solutions to the hypergeometric system *of differential equations.*

Proposition 6.5.17. *$\mathcal{S}(\lambda, k)$ is also a sheaf of $H(R^+, k)$-modules.*

Proof. $\mathcal{S}(\lambda, k)$ is a presheaf because it is the composition of the presheaf $\mathcal{O}$, the restriction functor from the category of $\mathfrak{D}R_{\mathfrak{R}}$-modules to the category of $H(R^+, k)$-modules, and the isotypic component functor (cf. Proposition 2.3.11).

Note that $\mathcal{S}(\lambda, k)$ is a presheaf of subspaces of $\mathcal{O}$, with the restrictions coming from $\mathcal{O}$. Thus, the identity property is given, and we just need to show that the gluing of sections of $\mathcal{S}(\lambda, k)$ again is a section of $\mathcal{S}(\lambda, k)$. Let $(U_i)_{i \in I}$ be a W-invariant cover of $U \subseteq H^{reg}$, let $f \in \mathcal{O}(U)$ with $f|_{U_i} \in \mathcal{S}(\lambda, k)(U_i)$ for all $i \in I$. Let $p \in S(\mathfrak{h})^W$, then

$$(p \cdot f)|_{U_i} = p \cdot f|_{U_i} = p \cdot f_i \in p(\lambda) f_i = (p(\lambda) f)|_{U_i}$$

(because restriction is $H(R^+, k)$-linear), which holds for every $i \in I$. By the identity axiom, this shows $p \cdot f = p(\lambda) f$, hence $f \in \mathcal{S}(\lambda, k)(U)$. Thus the gluing axiom is satisfied. □

Theorem 6.5.18. *For any choice of λ, k, the sheaf $\mathcal{S}(\lambda, k)$ is locally constant with the stalks of $\mathcal{S}(\lambda, k)^\lambda$ having $\mathbb{C}$-dimension $\#W$.*

Sketch of Proof. See [HS94, theorem 4.1.8] and [Opd00, section 7.2].

The idea is that the equations $p \cdot f = p(\lambda)f$ for $p \in S(\mathfrak{h})$ are equivalent to $(T_\xi(k) - \lambda(\xi)) \cdot f = 0$ for $\xi \in \mathfrak{h}$. By identitfying section of $\mathcal{O}$ with sections of a $\#W$-dimensional vector bundle over H^{reg}/W, we can build a connection out of $T_\xi(k) - \lambda(\xi)$: note that for $f \in \mathcal{O}(U)^W$ and $g \in \mathcal{O}(U)$ we have

$$(T_\xi(k) - \lambda(\xi))(fg) = \partial_\xi f g + f(T_\xi(k) - \lambda(\xi))g.$$

This connection is called the *Knizhnik–Zamolodchikov* (KZ) connection. Since the $T_\xi(k)$ commute with each other, the KZ connection is flat and therefore has flat sections. Flatness of a section f directly translates to $T_\xi(k)f = \lambda(\xi)f$ for all $\xi \in \mathfrak{h}$, and therefore to $f \in \mathcal{S}(\lambda, k)(U)^\lambda$.

Since the vector bundle that $\mathcal{O}(U)$ relates to is $\#W$-dimensional, there are $\#W$ linearly independent elements in $\mathcal{S}(\lambda, k)(U)^\lambda$ for U small enough. Local constancy follows from the fact that for simply connected neighbourhoods (in H^{reg}/W) we can (uniquely) define flat sections by parallel transport. □

Corollary 6.5.19. *Let $\lambda \in \mathfrak{h}$ be such that $\mu(\alpha_i^\vee) \neq 0, \pm k_i$ for $\mu \in W\lambda, i = 1, \ldots, n$. Let $p \in H^{reg}$, then*

$$\mathcal{O}_p[\chi_\lambda] \cong \mathcal{I}_\lambda^{\#W}$$

is the direct sum of $\#W$ copies of the simple $H(R^+, k)$-module $\mathcal{I}_\lambda$. (Here, $\mathcal{O}_p$ is the stalk at $p \in H^{reg}$.)

Proof. From Lemma 6.5.11 and Theorem 6.5.18. □

The notion of a locally constant sheaf of (finite-dimensional) vector spaces is a way to model the behaviour of multi-valued functions: functions that can be (uniquely) continued to always growing (simply connected) domains until topology "gets in the way". In [Pha05, chapter VIII] we see how the classical examples of complex multi-valued functions: roots/non-integer powers and the logarithm, and more generally solutions to complex ODEs with regular singular points can be described using locally constant sheaves.

Furthermore, as used in the proof of Theorem 6.5.18, flat sections of a vector bundle with flat connection also form a locally constant sheaf because we can use the parallel transport to continue flat sections to larger and larger simply connected domains (and flatness ensures that the parallel transport is independent of the path chosen).

A locally constant sheaf of finite-dimensional vector spaces over a manifold is also known as a *local system* and there are multiple equivalent ways of describing them:

- locally constant sheaf
- vector bundle with flat connection
- representation (of constant dimension) of the fundamental groupoid

- representation of the fundamental group (if the manifold is connected).

These last two representations are referred to as *monodromy* and one property of local systems is that a germ $s \in \mathcal{F}_p$ can be extended to a section over a connected neighbourhood $U \ni p$ (not necessarily simply connected) iff s is invariant under the monodromy representation of $\pi_1(U,p)$.

To find out if our local solutions to the hypergeometric system can be extended to interesting domains, we describe the monodromy representation. It turns out, it can be phrased in terms of representations of yet another Hecke algebra:

Definition 6.5.20. *Let $q_1, \dots, q_n$ be complex numbers, define the* finite-dimensional Hecke algebra *$H^{fin}(R^+, q)$ to be the $\mathbb{C}$-algebra generated by $T_1, \dots, T_n$ subject to*

$$T_i T_j \cdots = T_j T_i \cdots \qquad (m_{ij} \text{ factors}, i \neq j)$$

and $(T_i - 1)(T_i + q_i) = 0$. Here $m_{ij} = \alpha_i(\alpha_j^\vee)\alpha_j(\alpha_i^\vee)$, so that the T_i satisfy the same braid relations as the r_i, but don't square to 1 – just like the intertwiners.

Define furthermore the affine Hecke algebra *$H^{aff}(R^+, q)$ to be the tensor product $H^{fin}(R^+, q) \otimes \mathbb{C}[\mathbb{Z}R^\vee]$ (monomials from $v \in \mathbb{Z}R^\vee$ written as θ_v), subject to*

$$T_i \theta_v = \theta_{r_i(v)} T_i + (q_i - 1)\frac{\theta_v - \theta_{r_i(v)}}{1 - \theta_{-\alpha_i^\vee}}.$$

Theorem 6.5.21. *Assume that $\lambda(\alpha^\vee) \notin \mathbb{Z}$ for all $\alpha \in R$. Then the monodromy action on stalks of $\mathcal{S}(\lambda, k)$ induces a representation of $H^{aff}(R^+, q)$ where*

$$q_j = \exp\Big(-2\pi i\Big(k_{\frac{1}{2}\alpha_i} + k_{\alpha_i}\Big)\Big).$$

(Evidently, this action commutes with the action of $H(R^+, k)$.)

In particular, as $H^{aff}(R^+, q)$-modules, we have

$$\mathcal{S}(\lambda, k)_p \cong H^{aff}(R^+, q) \otimes_{\mathbb{C}[\mathbb{Z}R^\vee]} \mathbb{C}_{\exp(2\pi i\lambda)}$$

where $\mathbb{C}_{\exp(2\pi i\lambda)}$ is the 1-dimensional $\mathbb{C}[\mathbb{Z}R^\vee]$ module with $v \cdot 1 = \exp(2\pi i\lambda(v))$.

Proof. See [Opd00, theorem 6.8 and the following remarks] and [HS94, corollary 4.3.8].

That the monodromy action commutes with the action of $H(R^+, k)$ follows from the fact that $\mathcal{S}(\lambda, k)$ is a locally constant sheaf of $H(R^+, k)$-modules, so parallel transport is also $H(R^+, k)$-linear. □

Here, the T_i correspond to a path that connects p and $r_i(p)$ in H^{reg}, and the θ_v correspond to paths of the shape $t \mapsto p\exp(2\pi tv)$ (up to a numerical factor of $\exp(2\pi i\rho(k)(v))$). In other words: $H^{fin}(R^+, q)$ corresponds to the monodromy originating from the covering $H^{reg} \to H^{reg}/W$, and $\mathbb{C}[\mathbb{Z}R^\vee]$ corresponds to monodromy originating from the covering $\mathfrak{h}^{reg} \to H^{reg}$, i.e. to the compact part of the torus H.

The theorem shows that unless $\lambda \in \rho(k) + P^+$ (in which case we have already found the solutions: Jacobi polynomials), we can't even hope to find something that is invariant under all the monodromy. However, if we somewhat reduce our ambitions and don't try to extend our local solutions to H^{reg}, but only to a tubular neighbourhood U of

$$A^{reg} = \{\phi \in H^{reg} \mid \operatorname{im}(\phi) \subseteq \mathbb{R}_{>0}\},$$

we don't have to find invariants under the whole fundamental group, but just invariants under the fundamental group of U (equivalently: of A^{reg}). It just so happens that the monodromy representation of A^{reg} corresponds to a representation of $H^{fin}(R^+, q)$. In other words: we "cut open" the loops around the compact tori and then satisfy only the invariance requirement along the remaining loops.

As it turns out, this approach is successful and we have:

Theorem 6.5.22. *Let $U \subseteq H^{reg}$ be a tubular neighbourhood of*

$$A^{reg} = \{\phi \in H^{reg} \mid \operatorname{im}(\phi) \subseteq \mathbb{R}_{>0}\},$$

then there exists up to scaling a unique section $F(\lambda, k) \in \mathcal{S}(\lambda, k)(U)^W$.

Proof. See [Opd00, theorem 6.13]. □

Corollary 6.5.23. *Let $U \subseteq H^{reg}$ be a tubular neighbourhood of A^{reg} (or any other maximally noncompact real form), and let $\lambda \in \mathfrak{h}$ be such that $\mu(\alpha_i^\vee) \neq 0, \pm k_i$ ($\mu \in W\lambda, i = 1, \ldots, n$), then*

$$\mathcal{O}(U)[\chi_\lambda] \cong \mathcal{I}_\lambda.$$

Proof. Follows from Lemma 6.5.11 and Theorem 6.5.22. □

Definition 6.5.24. *In case this unique function is nonzero at 1, we can without loss of generality choose it to satisfy $F(\lambda, k; 1) = 1$. This (now truly unique) function is called* (symmetric) Heckman–Opdam hypergeometric function*.*

Now, instead of focussing on $\mathbb{C}[W] \subseteq H(R^+, k)$-invariant, $H^{fin}(R^+, q)$-invariant sections of $\mathcal{S}(\lambda, k)$, we could also focus on weight vectors:

Corollary 6.5.25. *Let $U \subseteq H^{reg}$ be a tubular neighbourhood of A^{reg}. If either $\mu(\alpha_i^\vee) \neq 0, \pm k_i$ ($\mu \in W\lambda, i = 1, \ldots, n$) or* $\operatorname{Re}(k_\alpha) \geq 0$ *for one $\alpha \in R^+$, then there exists up to scaling a unique section $G(\lambda, k) \in \mathcal{S}(\lambda, k)(U)^\lambda$.*

Proof. For λ generic, this follows from Corollary 6.5.23 and the fact that the weight spaces of the $\mathcal{I}_\lambda$ have dimension 1 for λ generic.

For λ is not generic, but we have $\operatorname{Re}(k_\alpha) \geq 0$, it is possible to show that the singularities are removable ([Opd00, lemma 7.7]). □

Definition 6.5.26. *In case this unique function is nonzero at 1, we can without loss of generality choose it to satisfy $G(\lambda, k; 1) = 1$. This function is called the* non-symmetric Heckman–Opdam hypergeometric function*.*

6.6. Hypergeometric System

In the last section we have seen what the χ_λ-isotypic components $\mathcal{S}(\lambda,k)(U), \mathcal{S}(\lambda,k)_p$ of the $H(R^+,k)$-modules $\mathcal{O}(U), \mathcal{O}_p$ look like, and thereby proved results about the hypergeometric system of differential equations. This leaves one major question about this whole subject unanswered: what is this system when expressed in coordinates, and what does it have to do with the differential operator from 5.1.4? Let's now answer that. Along the way we'll also see why the system is called *hypergeometric*.

Definition 6.6.1. *Let $\xi_1, \dots \xi_n \in \mathfrak{h}$ be an orthonormal basis, let (π, V) be a representation of W. Let k be a multiplicity vector. Then define*

$$ML^\pi(k) := \beta_\pi(T_{\xi_i}(k)^2) \in \mathbb{D}_{\mathfrak{R}}.$$

In particular, if V is 1-dimensional and $f \in \mathcal{O}(U)$ transforms like an element of V, then $ML^\pi(k)f = \sum_{i=1}^n \xi_i^2 \cdot f$.

For (π, V) being the trivial representation, we just write $ML(k)$. This is called the quadratic hypergeometric differential operator*.*

Proposition 6.6.2. *We have*

$$ML(k) = \Delta + \sum_{\alpha \geq 0} k_\alpha \frac{1 + e(-\alpha)}{1 - e(-\alpha)} \partial_{X_\alpha} + \|\rho(k)\|^2$$

Proof. We have

$$\begin{aligned}
\sum_{i=1}^n T_{\xi_i}(k)^2 &= \sum_{i=1}^n \left(\partial_{\xi_i} + \sum_{\alpha \geq 0} k_\alpha \alpha(\xi_i) \Delta_\alpha - \rho(k)(\xi_i) \right)^2 \\
&= \sum_{i=1}^n \partial_{\xi_i}^2 + \frac{1}{2} \sum_{i=1}^n \sum_{\alpha,\beta \geq 0} k_\alpha k_\beta \alpha(\xi_i) \beta(\xi_i) \{\Delta_\alpha, \Delta_\beta\} + \sum_{i=1}^n \rho(k)(\xi_i)^2 \\
&\quad + \sum_{i=1}^n \sum_{\alpha \geq 0} k_\alpha \alpha(\xi_i) \{\partial_{\xi_i}, \Delta_\alpha\} \\
&\quad - 2 \sum_{i=1}^n \rho(k)(\xi_i) \partial_{\xi_i} - 2 \sum_{i=1}^n \sum_{\alpha \geq 0} k_\alpha \alpha(\xi_i) \rho(k)(\xi_i) \Delta_\alpha \\
&= \Delta + \frac{1}{4} \sum_{\alpha,\beta} k_\alpha k_\beta (1 + 2\{\Delta_\alpha, \Delta_\beta\} - 2\Delta_\alpha - 2\Delta_\beta) \\
&\quad + \sum_{\alpha \geq 0} k_\alpha \{\partial_{X_\alpha}, \Delta_\alpha\} - 2\partial_{X_{\rho(k)}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\{\partial_{X_\alpha}, \Delta_\alpha\} &= \frac{1}{1 - e(-\alpha)} \partial_{X_\alpha} (1 - r_\alpha) + \frac{1}{1 - e(-\alpha)} \partial_{X_\alpha} (1 + r_\alpha) - \frac{\|\alpha\|^2}{(1 - e(-\alpha))^2} (1 - r_\alpha) \\
&= \frac{2}{1 - e(-\alpha)} \partial_{X_\alpha} - \frac{\|\alpha\|^2}{1 - e(-\alpha)} \Delta_\alpha.
\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{n} T_{\xi_i}(k)^2 &= \Delta + \sum_{\alpha\geq 0} k_\alpha \frac{1+e(-\alpha)}{1-e(-\alpha)}\partial_{X_\alpha} \\ &\quad + \frac{1}{4}\sum_{\alpha,\beta\geq 0} k_\alpha k_\beta \langle\alpha,\beta\rangle(1+2\{\Delta_\alpha,\Delta_\beta\}-2\Delta_\alpha-2\Delta_\beta) \\ &\quad - \sum_{\alpha\geq 0} k_\alpha \frac{\|\alpha\|^2}{1-e(-\alpha)}\Delta_\alpha \\ &= \Delta + \sum_{\alpha\geq 0} k_\alpha \frac{1+e(-\alpha)}{1-e(-\alpha)}\partial_{X_\alpha} + \frac{1}{8}\sum_{\alpha,\beta\geq 0} k_\alpha k_\beta\{2\Delta_\alpha - 1, 2\Delta_\beta - 1\} \\ &\quad - \sum_{\alpha\geq 0} k_\alpha \frac{\|\alpha\|^2}{1-e(-\alpha)}\Delta_\alpha.\end{aligned}$$

On a symmetric function, Δ_α acts like 0, so all terms that have a Δ_α on the right vanish when applying β_π (π the trivial representation). Thus, we are left with

$$\Delta + \sum_{\alpha\geq 0} k_\alpha \frac{1+e(-\alpha)}{1-e(-\alpha)}\partial_{X_\alpha} + \frac{1}{4}\sum_{\alpha,\beta\geq 0} k_\alpha k_\beta \langle\alpha,\beta\rangle \qquad \square$$

Example 6.6.3 (Type A_n)**.** *Take*

$$\mathfrak{h} = \left\{(\chi_0,\dots,\chi_n)\in\mathbb{C}^{n+1} \;\middle|\; \chi_0+\dots+\chi_n = 0\right\}$$

with the inner product inherited from $\mathbb{C}^{n+1}$*, and*

$$H = \left\{(\rho_0,\dots,\rho_n)\in(\mathbb{C}^\times)^{n+1} \;\middle|\; \rho_0\cdots\rho_n = 1\right\}.$$

Then we can choose the positive roots to be $e_i^* - e_j^*$ $(i>j)$*, then*

$$(e_i^*-e_j^*)^\vee = 2\frac{e_i-e_j}{2} = e_i - e_j$$

and $e(e_i^*-e_j^*)(\rho_0,\dots,\rho_n) = \frac{\rho_i}{\rho_j}$*.*

There is only one Weyl orbit, so the multiplicity vector is just the number k*. In particular,*

$$\rho(k) = \frac{k}{2}\sum_{i>j}(e_i-e_j) = \frac{k}{2}\sum_{i=0}^{n}(2i-n)e_i,$$

hence

$$\begin{aligned}\|\rho(k)\|^2 &= k^2\sum_{i=0}^{n}\left(i^2 - in + \frac{n^2}{4}\right) \\ &= k^2\left((n+1)\frac{n^2}{4} - n\frac{n(n+1)}{2} + \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \\ &= \frac{k^2}{4}\frac{n(n+1)(n+2)}{3}.\end{aligned}$$

Then,

$$ML(k) = \Delta + k\sum_{i>j} \frac{\rho_i+\rho_j}{\rho_i-\rho_j}(\rho_i\partial_{\rho_i} - \rho_j\partial_{\rho_j}) + \frac{k^2}{4}\frac{n(n+1)(n+2)}{3},$$

or in terms of logarithmic coordinates $\chi_0, \ldots, \chi_n$:

$$\Delta + k\sum_{i>j} \coth\left(\frac{\chi_i-\chi_j}{2}\right)(\partial_{\chi_i} - \partial_{\chi_j}) + \frac{k^2}{4}\frac{n(n+1)(n+2)}{3}$$

Example 6.6.4 (Type BC_n)**.** *Take $\mathfrak{h} = \mathbb{C}^n$ with coordinates $\chi_1, \ldots, \chi_n$ with inner product inherited from $\mathbb{C}^n$, and $H = (\mathbb{C}^\times)^n$ with coordinates $\rho_1, \ldots, \rho_n$. Then we can choose the positive roots to be*

$$e_i^*, 2e_i^*, e_i^* - e_j^*, e_i^* + e_j^* \qquad (i > j).$$

Then there are three Weyl orbits: short, medium, and long roots, labelled with k_1, k_2, k_3, respectively, and

$$\begin{aligned} ML(k) = \Delta &+ \sum_{i=1}^n \left(k_1\frac{\rho_i+1}{\rho_i-1} + 2k_3\frac{\rho_i+\rho_i^{-1}}{\rho_i-\rho_i^{-1}}\right)\rho_i\partial_{\rho_i} \\ &+ \sum_{i>j} k_2\left(\frac{\rho_i+\rho_j}{\rho_i-\rho_j}(\rho_i\partial_{\rho_i} - \rho_j\partial_{\rho_j}) + \frac{\rho_i\rho_j+1}{\rho_i\rho_j-1}(\rho_i\partial_{\rho_i} + \rho_j\partial_{\rho_j})\right) \\ &+ \|\rho(k)\|^2 \\ &= \Delta + \sum_{i=1}^n \frac{2k_1 + (k_1+2k_3)(\rho_i+\rho_i^{-1})}{\rho_i-\rho_i^{-1}}\rho_i\partial_{\rho_i} + 2k_2\sum_{i>j}\frac{\rho_i-\rho_i^{-1}}{\rho_i+\rho_i^{-1}-\rho_j-\rho_j^{-1}}\rho_i\partial_i \\ &- 2k_2\sum_{i>j}\frac{\rho_j-\rho_j^{-1}}{\rho_i+\rho_i^{-1}-\rho_j-\rho_j^{-1}}\rho_j\partial_{\rho_j} + \|\rho(k)\|^2 \end{aligned}$$

in ρ_i coordinates and

$$\begin{aligned} ML(k) &= \Delta + \sum_{i=1}^n (k_1 \operatorname{csch}(\chi_i) + (k_1+2k_3)\coth(\chi_i))\partial_{\chi_i} \\ &\quad + 2k_2\sum_{i>j}\frac{\sinh(\chi_i)\partial_{\chi_i} - \sinh(\chi_j)\partial_{\chi_j}}{\cosh(\chi_i)-\cosh(\chi_j)} + \|\rho(k)\|^2 \\ &= \Delta + \sum_{i=1}^n \left(k_1\coth\left(\frac{\chi_i}{2}\right) + 2k_3\coth(\chi_i)\right)\partial_{\chi_i} \\ &\quad + k_2\sum_{i>j}\left(\coth\left(\frac{\chi_i-\chi_j}{2}\right)(\partial_{\chi_i}-\partial_{\chi_j}) + \coth\left(\frac{\chi_i+\chi_j}{2}\right)(\partial_{\chi_i}+\partial_{\chi_j})\right) \\ &\quad + \|\rho(k)\|^2 \end{aligned}$$

Lemma 6.6.5. *In the general setting define*

$$\delta(k) := \prod_{\alpha\in R^+}\left(e\left(\frac{\alpha}{2}\right) - e\left(-\frac{\alpha}{2}\right)\right)^{k_\alpha}.$$

Then $\delta(k)ML(k)\delta(k)^{-1} = \Delta + V$ *where*

$$V = \sum_{\alpha\in R^+}\frac{k_\alpha(1-k_\alpha-2k_{2\alpha})\|\alpha\|^2}{\left(e(\frac{\alpha}{2})-e(-\frac{\alpha}{2})\right)^2}.$$

Proof. First note that for an entire function $h \in \mathcal{O}(H^{reg})$ that's nonzero everywhere and a differential operator D, the operator $h^\alpha D h^\beta$ is always a (globally defined) differential operator if $\alpha + \beta \in \mathbb{Z}$, regardless of which domain we define our logarithm and hence our power functions in. This follows from induction in the degree of D:
"degree 0": $h^\alpha f h^\beta = f h^{\alpha+\beta}$.
"$d \to d+1$": Let $\xi \in \mathfrak{h}$, then

$$\begin{aligned} h^\alpha D\xi h^\beta &= h^\alpha D h^{-\alpha} h^\alpha \xi h^\beta \\ &= h^\alpha D h^{-\alpha} h^{\alpha+\beta}\xi + h^\alpha D h^{-\alpha} h^{\alpha+\beta-1}\partial_\xi(h). \end{aligned}$$

By induction hypothesis, $h^\alpha D h^{-\alpha}$ is a (globally defined) differential operator. Furthermore, so are $h^{\alpha+\beta}\xi$ and $h^{\alpha+\beta-1}\partial_\xi(h)$.

The calculation itself can be found at [HS94, theorem 2.1.1] (and the expression is also valid for our Dunkl operators as per [Opd00, example 6.2]). Note that the authors of this book call $\delta^{1/2}$ what we call δ. □

As a consequence, a function $f \in \mathcal{O}(U)$ satisfies $ML(k)f = \|\lambda\|^2 f$ iff

$$\|\lambda\|^2\delta(k)f = \delta(k)ML(k)\delta(k)^{-1}\delta(k)f = (\Delta + V)\delta(k)f,$$

which is (up to constants) the time-independent Schrödinger equation for the *Calogero–Sutherland model* (for the root system R) from the introduction. As was already hinted there, for A_n and BC_n we get potentials with particularly nice interpretations:

Example 6.6.6 (Type A_n)**.** *Here we have*

$$\begin{aligned} V &= 2k(1-k)\sum_{i>j}\frac{1}{\rho_i\rho_j^{-1} + \rho_i^{-1}\rho_j - 2} \\ &= \frac{k(1-k)}{2}\sum_{i>j}\operatorname{csch}^2\left(\frac{\chi_i - \chi_j}{2}\right), \end{aligned}$$

i.e. a potential of $n+1$ *1-dimensional particles interacting via a* csch^2 *potential.*

Example 6.6.7 (Type BC_n)**.** *Here we have*

$$\begin{aligned} V &= \sum_{i=1}^{n} \left(\frac{k_1(1-k_1-2k_3)}{\rho_i + \rho_i^{-1} - 2} + \frac{4k_3(1-k_3)}{\rho_i^2 + \rho_i^{-2} - 2} \right) \\ &\quad + \sum_{i>j} 2k_2(1-k_2) \left(\frac{1}{\rho_i \rho_j^{-1} + \rho_i^{-1}\rho_j - 2} + \frac{1}{\rho_i\rho_j + \rho_i^{-1}\rho_j^{-1} - 2} \right) \\ &= \sum_{i=1}^{n} \left(\frac{k_1}{2}\left(\frac{1-k_1}{2} - k_3\right) \operatorname{csch}^2\left(\frac{\chi_i}{2}\right) + k_3(1-k_3)\operatorname{csch}^2(\chi_i) \right) \\ &\quad + \sum_{i>j} \frac{k_2}{2}(1-k_2)\left(\operatorname{csch}^2\left(\frac{\chi_i - \chi_j}{2}\right) + \operatorname{csch}^2\left(\frac{\chi_i + \chi_j}{2}\right) \right), \end{aligned}$$

the potential of n 1-dimensional particles, each of which in a Pöschl–Teller potential, interacting with each other and with their reflections via csch^2*.*

6.7. Connection to Scalar Conformal Blocks

The attentive reader may have noticed that the operator $ML(k)$ for BC_n (as given in Example 6.6.4) look oddly familiar. Especially, if we choose $n = 2$, we obtain

$$\begin{aligned} &\partial_{\chi_1}^2 + \partial_{\chi_2}^2 + \sum_{i=1}^{2} \left(k_1 \coth\left(\frac{\chi_i}{2}\right) + 2k_3 \coth(\chi_i) \right) \partial_{\chi_i} \\ &+ k_2 \left(\coth\left(\frac{\chi_1 - \chi_2}{2}\right)(\partial_{\chi_1} - \partial_{\chi_2}) + \coth\left(\frac{\chi_1 + \chi_2}{2}\right)(\partial_{\chi_1} + \partial_{\chi_2}) \right) \\ &+ \left(\frac{k_1}{2} + k_3\right)^2 + \left(\frac{k_1}{2} + k_2 + k_3\right)^2. \end{aligned}$$

Written this way, it has a lot of similarity with the expression (5.2) we got for the Casimir equation for scalar conformal blocks, expanded in (χ_1, χ_2)-coordinates. In particular this Casimir differential operator is

$$ML(k) - \frac{1}{4} - \left(k_2 + \frac{1}{2}\right)^2$$

for k being related to the dimension d and the parameters Δ_1, Δ_2 of the representation V as follows:

$$k = \left(\Delta_1, \frac{d-2}{2}, \frac{1 - \Delta_1 - \Delta_2}{2}\right).$$

In particular, a conformal block f is said to have parameters Δ, ℓ if it satisfies

$$\Omega_{\mathfrak{g}} \cdot f = \frac{1}{2}(\Delta(\Delta - d) + \ell(\ell + d - 2))f.$$

Such an f then satisfies

$$\begin{aligned}
2ML(k)f &= \Delta(\Delta-d)f + \ell(\ell+d-2)f + \frac{f}{2} + 2\Big(k_2+\frac{1}{2}\Big)^2 f \\
&= \Delta(\Delta-2k_2-2)f + \ell(\ell+2k_2)f + 2k_2^2 f + 2k_2 f + f \\
&= \Big(\Delta^2+k_2^2+1-2\Delta k_2-2\Delta+2k_2\Big)f + \Big(\ell^2+2k_2\ell+k_2^2\Big)f \\
&= \Big((\Delta+k_2+1)^2+(\ell+k_2)^2\Big)f \\
ML(k)f &= \left\|\frac{1}{\sqrt{2}}\begin{pmatrix}\Delta+k_2+1\\ \ell+k_2\end{pmatrix}\right\|^2 f.
\end{aligned}$$

To conclude this section, let us find out why the hypergeometric system of differential equations is called hypergeometric. For this, let us consider the BC_1 case. In BC_1 there are two Weyl orbits: short and long, labelled with k_1, k_2, respectively. We have

$$\begin{aligned}
ML(k) &= (\rho\partial_\rho)^2 + \left(k_1\frac{\rho+1}{\rho-1} + 2k_2\frac{\rho+\rho^{-1}}{\rho-\rho^{-1}}\right)\rho\partial_\rho + \left(\frac{k_1}{2}+k_2\right)^2 \\
&= \partial_\chi^2 + \left(k_1\coth\Big(\frac{\chi}{2}\Big) + 2k_2\coth(\chi)\right)\partial_\chi + \left(\frac{k_1}{2}+k_2\right)^2.
\end{aligned}$$

Now we go to the z-coordinate from Section 5.3, i.e. $z = \frac{1}{2} - \frac{1}{4}(\rho+\rho^{-1}) = -\sinh^2(\frac{\chi}{2})$. We obtain

$$ML(k) = -z(1-z)\partial_z^2 - \left(k_1+k_2+\frac{1}{2}-(k_1+2k_2+1)z\right)\partial_z + \left(\frac{k_1+2k_2}{2}\right)^2.$$

The equation $ML(k)f = \lambda^2 f$ becomes

$$\begin{aligned}
0 &= -\left(z(1-z)\partial_z^2 + \left(k_1+k_2+\frac{1}{2}-(k_1+2k_2+1)z\right)\partial_z - \left(\frac{k_1+2k_2}{2}\right)^2 + \lambda^2\right)f \\
&= -\Big(z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab\Big)f,
\end{aligned}$$

where we defined

$$c := k_1+k_2+\frac{1}{2}, \qquad a,b = \frac{k_1+2k_2}{2} \pm \lambda.$$

This is precisely the hypergeometric ODE, and since the correspondence $k_1, k_2, \lambda \leftrightarrow a, b, c$ is bijective, the quadratic hypergeometric equation from the theory of Dunkl operators for BC_1 serves exactly the same purpose as the hypergeometric ODE.

6.8. Connection to Spinorial Conformal Blocks

Let us now introduce the second method to deal with the fact that in $H(R^+, k)$ we generally have $wpw^{-1} \neq w(p)$ for $w \in W, p \in S(\mathfrak{h})$. In Section 6.5.2, we introduced intertwiners to find replacements for the w's that interact nicely with the Dunkl operators. We could, however, go the other way around and use the Dunkl operators defined in [HS94].

6.8.1. Heckman–Dunkl Operators

Let $\xi \in \mathfrak{h}$, then define

$$S_\xi(k) = T_\xi(k) + \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha(\xi) r_\alpha = \partial_\xi + \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha(\xi) \frac{1 + e(-\alpha)}{1 - e(-\alpha)} (1 - r_\alpha), \tag{6.2}$$

or in terms of the Hecke algebra: $s_\xi(k) := \xi + u_\xi(k)$ where

$$u_\xi(k) = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha(\xi) r_\alpha.$$

From (6.2) we can see that replacing α by $-\alpha$ in the sum doesn't change the sum, so that $S_\xi(k)$ is independent from the choice of positive roots, which shows that $wS_\xi(k)w^{-1} = S_{w(\xi)}(k)$. On the flip side, the S_ξ now have nontrivial commutators, so by switching from one flavour of Dunkl operator to the other, we have sacrificed commutativity for a nice W-action.

6.8.2. A Closer Look at BC_2

In order to find a connection to the expressions for the Dirac operator (or in particular, the differential operators it can be written in terms of) we now have a more in-depth look at BC_2, in particular at its Weyl group. Recall that BC_2 is made up of the following roots:

$$e_1^*, 2e_1^*, e_2^*, 2e_2^*, e_1^* + e_2^*, e_1^* - e_2^*$$

and their negatives. We can choose the above roots to be the positive roots, in which case the simple roots are given by $\alpha := e_1^*$ and $\beta := e_2^* - e_1$. The corresponding reflections r_α, r_β then generate the Weyl group.

W has five conjugacy classes of elements:

$$\{1\}, \{r_\alpha, r_\beta r_\alpha r_\beta\}, \{r_\beta, r_\alpha r_\beta r_\alpha\}, \{r_\alpha r_\beta, r_\beta r_\alpha\}, \{r_\alpha r_\beta r_\alpha r_\beta\}$$

(the identity, reflections along the axes, reflections along the diagonals, rotations by $\frac{\pi}{2}$, and rotation by π). By classical finite group representation theory, these conjugacy classes correspond to five irreducible representations. Since $\#W = 8$, exactly one of these representations has to be > 1-dimensional, namely 2-dimensional. The 1-dimensional representations correspond to the (multiplicative) characters $\chi_{(s,t)}$ given by $\chi_{(s,t)}(r_\alpha) = s$ and $\chi_{(s,t)}(r_\beta) = t$ for $s, t \in \{\pm 1\}$, and the 2-dimensional representation is the representation of W on $\mathfrak{h}$, which has character

$$\chi_2(1) = 2, \chi_2(r_\alpha) = \chi_2(r_\beta) = \chi_2(r_\alpha r_\beta) = 0, \chi_2(r_\alpha r_\beta r_\alpha r_\beta) = -2.$$

6.8.3. Matching Differential Operators

In the 1d representation (s,t) we obtain the following S-Dunkl operators (write S_i for $S_{e_i}(k)$):

$$\begin{aligned}\beta_{(s,t)}(S_1) = \partial_{\chi_1} + \frac{1}{2}\Bigg(&\Big(k_1 \coth\Big(\frac{\chi_1}{2}\Big) + 2k_3 \coth(\chi_1)\Big)(1-s)\\ &+ k_2\Big(\coth\Big(\frac{\chi_1+\chi_2}{2}\Big) + \coth\Big(\frac{\chi_1-\chi_2}{2}\Big)\Big)(1-t)\Bigg)\\ \beta_{(s,t)}(S_2) = \partial_{\chi_2} + \frac{1}{2}\Bigg(&\Big(k_1 \coth\Big(\frac{\chi_2}{2}\Big) + 2k_3 \coth(\chi_2)\Big)(1-s)\\ &+ k_2\Big(\coth\Big(\frac{\chi_1+\chi_2}{2}\Big) - \coth\Big(\frac{\chi_1-\chi_2}{2}\Big)\Big)(1-t)\Bigg).\end{aligned}$$

We will now look at $\mathbb{C}[H^{reg}]$-linear combinations of these operators and try to find the differential operators obtained in Section 5.3.3 (in χ-coordinates). We have

$$\begin{aligned}&\coth\Big(\frac{\chi_1}{2}\Big)\beta_{(s,t)}(S_1) \pm \coth\Big(\frac{\chi_2}{2}\Big)\beta_{(s,t)}(S_2)\\ =&\coth\Big(\frac{\chi_1}{2}\Big)\partial_{\chi_1} \pm \coth\Big(\frac{\chi_2}{2}\Big)\partial_{\chi_2}\\ &+\frac{1}{2}\Big((k_1+k_3)\coth^2\Big(\frac{\chi_1}{2}\Big) \pm (k_1+k_3)\coth^2\Big(\frac{\chi_2}{2}\Big) + (1\pm 1)k_3\Big)(1-s)\\ &+\frac{k_2}{2}\frac{\cosh^2(\frac{\chi_1}{2}) \mp \cosh^2(\frac{\chi_2}{2})}{\sinh\big(\frac{\chi_1+\chi_2}{2}\big)\sinh\big(\frac{\chi_1-\chi_2}{2}\big)}(1-t).\end{aligned}$$

If we set $s=1$ and look at the $+$ case, we obtain

$$\coth\Big(\frac{\chi_1}{2}\Big)\partial_{\chi_1} + \coth\Big(\frac{\chi_2}{2}\Big)\partial_{\chi_2} + k_2(1-t) = E - 2k_3 - k_2 t,$$

and for the $-$ case we obtain

$$\coth\Big(\frac{\chi_1}{2}\Big)\partial_{\chi_1} - \coth\Big(\frac{\chi_2}{2}\Big)\partial_{\chi_2} + k_2(1-t)\frac{\cosh^2(\frac{\chi_1}{2}) + \cosh^2(\frac{\chi_2}{2})}{\sinh\big(\frac{\chi_1-\chi_2}{2}\big)\sinh\big(\frac{\chi_1+\chi_2}{2}\big)} = F^{-t}.$$

Conversely, we have

$$\begin{aligned}&\tanh\Big(\frac{\chi_1}{2}\Big)\beta_{(s,t)}(S_1) \pm \tanh\Big(\frac{\chi_2}{2}\Big)\beta_{(s,t)}(S_2)\\ =&\tanh\Big(\frac{\chi_1}{2}\Big)\partial_{\chi_1} \pm \tanh\Big(\frac{\chi_2}{2}\Big)\partial_{\chi_2}\\ &+\frac{1}{2}\Big((1\pm 1)(k_1+k_3) + k_3\tanh^2\Big(\frac{\chi_1}{2}\Big) \pm k_3\tanh^2\Big(\frac{\chi_2}{2}\Big)\Big)(1-s)\\ &+k_2(1-t)\frac{\sinh^2(\frac{\chi_1}{2}) \mp \sinh^2(\frac{\chi_2}{2})}{\sinh\big(\frac{\chi_1-\chi_2}{2}\big)\sinh\big(\frac{\chi_1+\chi_2}{2}\big)}.\end{aligned}$$

For $s=1$ we obtain for the $+$ case

$$\tanh\left(\frac{\chi_1}{2}\right)\partial_{\chi_1}+\tanh\left(\frac{\chi_2}{2}\right)\partial_{\chi_2}+k_2(1-t)=-G+2(k_1+k_2+k_3)-k_2t$$

and for the $-$ case

$$\tanh\left(\frac{\chi_1}{2}\right)\partial_{\chi_1}-\tanh\left(\frac{\chi_2}{2}\right)\partial_{\chi_2}+k_2(1-t)\frac{\sinh^2(\frac{\chi_1}{2})+\sinh^2(\frac{\chi_2}{2})}{\sinh\left(\frac{\chi_1-\chi_2}{2}\right)\sinh\left(\frac{\chi_1+\chi_2}{2}\right)}=-H^{-t}$$

if $w_\mu w_\nu=1$. This can be achieved in most practical cases, e.g. by choosing

$$w=\operatorname{diag}(-1,1\dots,1,-1,1),\qquad w=\operatorname{diag}(1,-1,-1,1,\dots,1,-1,-1)$$

for $q=0,1$, respectively.

To summarise, we can write the operators E,F,G,H used in (5.1) to express the Dirac operator as a matrix in terms of Dunkl (-derived differential) operators as

$$\begin{aligned}
E&=\coth\left(\frac{\chi_1}{2}\right)\beta_{(+,t)}(S_1)+\coth\left(\frac{\chi_2}{2}\right)\beta_{(+,t)}(S_2)+2k_3+k_2t\\
G&=-\tanh\left(\frac{\chi_1}{2}\right)\beta_{(+,t)}(S_1)-\tanh\left(\frac{\chi_2}{2}\right)\beta_{(+,t)}(S_2)+2(k_1+k_2+k_3)-k_2t\\
F^t&=\coth\left(\frac{\chi_1}{2}\right)\beta_{(+,-t)}(S_1)-\coth\left(\frac{\chi_2}{2}\right)\beta_{(+,-t)}(S_2)\\
H^t&=-\tanh\left(\frac{\chi_1}{2}\right)\beta_{(+,-t)}(S_1)+\tanh\left(\frac{\chi_2}{2}\right)\beta_{(+,-t)}(S_2).
\end{aligned}$$

6.8.4. Matching Representations

In the previous subsection, we have established a correspondence between the differential operators occurring in the matrix expression (5.1) and linear combinations of (Heckman–)Dunkl operators acting on different 1d representations of W. However, especially in the case of E,G (and $t=+$), this correspondence is tenuous at best since in this case the Dunkl operators become just the derivatives themselves, so we could build any differential operator as a $\mathbb{C}[H^{reg}]$-linear combination of them. That this correspondence also works out for E,G in the $t=-$ case is already a hint that this correspondence is not just spurrious, as is the correspondence for F,H.

However, we are still dealing with occurrences of seemingly random instances of $\beta_{(+,+)},\beta_{(+,-)}$ and it would be nice to get rid of them, maybe by assigning consistent W-representations to the different basis vectors. In order to achieve that we have a closer look at how W and in particular its generators r_α,r_β act on $\mathfrak{h},H$, and interact with S_1,S_2 and their linear combinations.

r_α inverts the 1st coordinate, i.e. we have $r_\alpha(\chi_1)=-\chi_1$ and $r_\alpha(\chi_2)=\chi_2$. Similarly, r_β inverts the difference between the 1st and 2nd coordinate, hence $r_\beta(\chi_1)=\chi_2$ and vice-versa.

Proposition 6.8.1. *For $s, t \in \{\pm\}$, the operators*

$$I_1^+ := \coth\left(\frac{\chi_1}{2}\right)S_1 + \coth\left(\frac{\chi_2}{2}\right)S_2, \qquad I_2^+ := -\tanh\left(\frac{\chi_1}{2}\right)S_1 - \tanh\left(\frac{\chi_2}{2}\right)S_2$$

are intertwiners between (s, t) and (s, t), and the operators

$$I_1^- := \coth\left(\frac{\chi_1}{2}\right)S_1 - \coth\left(\frac{\chi_2}{2}\right)S_2, \qquad I_2^- := -\tanh\left(\frac{\chi_1}{2}\right)S_1 + \tanh\left(\frac{\chi_2}{2}\right)S_2$$

are intertwiners between the (s, t) and $(s, -t)$ isotypic components (with respect to W) of any representation they act on.

Proof. Note that

$$\begin{aligned} r_\alpha S_1 &= -S_1 r_\alpha \qquad & r_\beta S_1 &= S_2 r_\beta \\ r_\alpha S_2 &= S_2 r_\alpha \qquad & r_\beta S_2 &= S_1 r_\beta, \end{aligned}$$

so that we obtain

$$\begin{aligned} r_\alpha I_1^\pm &= \left(\coth\left(-\frac{\chi_1}{2}\right)(-S_1) \pm \coth\left(\frac{\chi_2}{2}\right)S_2\right)r_\alpha \\ &= I_1^\pm r_\alpha \\ r_\beta I_1^\pm &= \left(\coth\left(\frac{\chi_2}{2}\right)S_2 \pm \coth\left(\frac{\chi_1}{2}\right)S_1\right)r_\beta \\ &= \pm I_1^\pm r_\beta \end{aligned}$$

and similarly $r_\alpha I_2^\pm = I_2^\pm r_\alpha$ and $r_\beta I_2^\pm = \pm I_2^\pm r_\beta$. If f lies in the (s, t)-isotypic component, we have

$$\begin{aligned} r_\alpha I_{1/2}^\pm f &= I_{1/2}^\pm r_\alpha f = s I_{1/2}^\pm f \\ r_\beta I_{1/2}^\pm f &= \pm I_{1/2}^\pm r_\beta f = \pm t I_{1/2}^\pm f, \end{aligned}$$

so that $I_{1/2}^\pm f$ lies in the $(s, \pm t)$-isotypic component, as claimed. □

Theorem 6.8.2. *Let $f \in \mathrm{Ward}(S \otimes V)$ be such that*

$$f_{\sigma\tau} = \sum_{I \subseteq \{\mu,\nu\}} a_I v_I + \sum_{I \subseteq \{\mu,\nu\}} \tilde{a}_I v_I$$

where $a_I, \tilde{a}_I \in \mathcal{O}(U)[(+,+)]$ if $\nu \notin I$ and $a_I, \tilde{a}_I \in \mathcal{O}(U)[(+,-)]$ if $\nu \in I$. Here μ, ν are the indices chosen such that $\xi_{\sigma\tau}$ is constructed using P_μ, P_ν, and we assume that $w_\mu w_\nu = 1$. In the basis used for (5.1)*, we then obtain the following*

$$(\mathcal{D}f)_{\sigma\tau} = \frac{1}{\sqrt{2}}((D \oplus D) + (C^+ \oplus C^-))f_{\sigma\tau}$$

where

$$D = \begin{pmatrix} 0 & 0 & I_1^+ & \epsilon_F I_1^- \\ 0 & 0 & \epsilon_H I_2^- & -I_2^+ \\ -I_2^+ & -\epsilon_F I_1^- & 0 & 0 \\ -\epsilon_H I_2^- & I_1^+ & 0 & 0 \end{pmatrix} \qquad \epsilon_F := \frac{w_0 w_\nu}{\sqrt{-\eta_{\mu\mu}\eta_{\nu\nu}}}, \quad \epsilon_H := \eta_{\mu\mu}\epsilon_F$$

and

$$C^\pm = \begin{pmatrix} 0 & 0 & \pm k_2 + 2k_3 & 0 \\ 0 & 0 & 0 & -2(k_1+k_2+k_3) \mp k_2 \\ -2(k_1+k_2+k_3) \pm k_2 & 0 & 0 & 0 \\ 0 & \mp k_2 + 2k_3 & & \end{pmatrix}.$$

The coordinate functions of $(\mathcal{D}f)_{\sigma\tau}$ have the same W-transformation behaviour as we required from $f_{\sigma\tau}$.

Proof. The basis is chosen in such a way that within the subspaces spanned by the v's and the $\tilde{v}$'s respectively, the isotypes always alternate, starting from $(+,+)$ and $(+,-)$, respectively. Note that in the matrix $D^\pm$ there is always an $F^\pm$ in even-numbered columns and an $H^\mp$ in odd-numbered columns. Using our assignment of isotypes, those are then just the respective actions of I_1^- and I_2^-, respectively. If we replace E, G by I_1^+, I_2^+ and extract the constants, we obtain the expressions from the claim.

Since I^+'s always occur in places where row number and column number have the same parity, and the I^-'s always occur in places where these numbers have opposite parity, by Proposition 6.8.1, $\mathcal{D}$ doesn't change the assigned isotypes. □

And with that we have expressed the components of the Dirac operator using Dunkl operators and $\mathbb{C}[H^{reg}]$, so fundamentally in terms of $\mathbb{C}[H^{reg}] \otimes_{\mathfrak{R}} \mathbb{D}R_{\mathfrak{R}}$. Since $H(R^+, k)$'s embedding into $\mathbb{D}R_{\mathfrak{R}}$ has no way of touching "interesting" regular functions, there is (to the author's knowledge) no direct way of phrasing this in terms of representations of $H(R^+, k)$. However, [Opd00] enlarges the degenerate affine Hecke algebra to the *extended* degenerate *double*-affine Hecke-algebra (edDAHA) $H^e(R^+, k)$, which as a vector space is isomorphic to $\mathbb{C}[H] \otimes H(R^+, k)$, and which can be defined in very much the same way as $H(R^+, k)$, except using the *extended* Weyl group (that includes translations along P, hence *double*-affine) and the affine root system generated by R.

Covering the (representation) theory of edDAHAs in this thesis would go too far, but suffice it to say that they also have representations on $\mathbb{C}[H], \mathbb{C}[H^{reg}]$ and $\mathcal{O}(U)$ that now also include the operations of multiplying by elements of $\mathbb{C}[H]$. If we localise $H^e(R^+, k)$ in the multiplicative set generated by $1 - e(-\alpha)$ $(\alpha \in R^+)$, we obtain an algebra big enough to contain our intertwiners $I^\pm_{1/2}$.

6.8.5. Assignment of Isotypes

Finally, let us now show that the transformation behaviour required by Theorem 6.8.2 is satisfied by $f_{\sigma\tau}$ for $f \in \mathrm{Ward}(S \otimes V)$ and thereby shed some light on the naturality of such a requirement.

Lemma 6.8.3. *Let $f \in \mathrm{Ward}(S \otimes V)$, let $\sigma, \tau \in \{s, t\}$ and let μ, ν the indices used in $\xi_{\sigma\tau}$. Then the coordinate functions of $f_{\sigma\tau}$ satisfy the isotype conditions required in Theorem 6.8.2.*

Proof. Recall from Section 5.3 that b is proportional to z_1, z_2. There is nothing particularly harmful in allowing b to be negative as well, and thereby extending the domain to also allowing $\mathrm{Im}(z_1) < 0$ (if z_1 and z_2 are complex conjugates) or allowing $z_1 < z_2$. Then the domain of $f_{\sigma\tau}$ is a W-invariant subset of $\mathfrak{h} = \mathbb{C}^2$ and we can study W's action on it. Note in particular that r_α inverts ρ_1 but doesn't affect z_1, z_2 at all. Consequently, $r_\alpha f_{\sigma\tau} = f_{\sigma\tau}$, which is why $(+,+)$ and $(+,-)$ are the only irreducible representations occurring in Theorem 6.8.2.

Now to the action of r_β. Recall that it exchanges χ_1, χ_2, and therefore also all other coordinates: ρ_1, ρ_2 and z_1, z_2, before ultimately leaving u, v unchanged. As noted before, b is proportional to $z_1 - z_2$, so exchanging z_1, z_2 inverts b. If (a, b) correspond to z_1, z_2 we then have

$$r_\beta f_{\sigma\tau}(z_1, z_2) = f(\exp(P_\mu) w \exp(aP_\mu - bP_\nu)).$$

To bring the argument back to our fundamental domain (with $b > 0$), let $m \in M$ be such that $me_\mu = e_\mu$ and $me_\nu = -e_\nu$. The same is then also true for $c_w(m)e_\nu$ and we have

$$r_\beta f_{\sigma\tau}(z_1, z_2) = f(c_w(m)\xi_{\sigma\tau}(a, b)m^{-1}) = c_w(m) \cdot f_{\sigma\tau}(z_1, z_2) \cdot m^{-1}.$$

Now, m's action on the right is trivial, and for $c_w(m)$'s action on the left note that for $p \in \mathrm{C}\ell(Y)$ we have

$$\phi_Y(c_w(m))p \cdot v_\emptyset = \mathrm{Ad}(c_w(m))(p)\phi_Y(c(w)) \cdot v_\emptyset = \mathrm{Ad}(c_w(m))(p) \cdot v_\emptyset$$

and similarly for $\tilde{v}_\emptyset$ because

$$\phi_Y(c_w(m)) \cdot \tilde{v}_\emptyset = \det(c_w(m))\tilde{v}_\emptyset = \tilde{v}_\emptyset.$$

Since the adjoint action of $c_w(m)$ on monomials made up of $K_\mu, K_\nu, P_\mu, P_\nu$ will negate anything that has a K_ν or a P_ν in it, we have

$$c_w(m) \cdot v_I = \begin{cases} -v_I & \nu \in I \\ v_I & \nu \notin I \end{cases}, \qquad c_w(m) \cdot \tilde{v}_I = \begin{cases} -\tilde{v}_I & \nu \in I \\ \tilde{v}_I & \nu \notin I \end{cases}$$

for $I \subseteq \{\mu\nu\}$. Consequently, if we expand

$$f_{\sigma\tau} = \sum_{I \subseteq \{\mu,\nu\}} (a_I v_I + \tilde{a}_I \tilde{v}_I),$$

we get

$$r_\beta f_{\sigma\tau} = \sum_{I \subseteq \{\mu,\nu\}} (-1)^{\#(I \cap \{\nu\})}(a_I v_I + \tilde{a}_I \tilde{v}_I),$$

which is exactly the transformation behaviour required in Theorem 6.8.2. □

Thus we see the transformation behaviour under W is a tiny remainder of the $MA \times MA$-biequivariance that we hadn't (fully) divided out yet. We can therefore expect that if there is also a correspondence between non-scalar conformal blocks and a kind of matrix CS model, there would be more instances of that interaction between Weyl group action and residual biequivariance under discrete subgroups of $MA \times MA$ (modulo stabiliser of the fundamental domain).

7. Conclusion and Outlook

And with that we have achieved what we set out to do. We have

- introduced a representation-theoretic framework for talking about n-point functions and conformal blocks (Section 4),
- reproduced the correspondence between the (quadratic) Casimir equation of scalar conformal blocks and the (quadratic) hypergeometric equation for BC_2 (Section 6.7,
- established a general theory for actions of Clifford algebra-valued (invariant) differential operators on spinorial conformal blocks (Section 4.3,
- computed the action of Kostant's cubic Dirac operator on (the simplest kind of) spinorial conformal blocks (Section 5.2),
- pondered the question why the cubic Dirac operator is the correct operator to describe (Section 4.4),
- found Dunkl operators lurking in the matrix entries of the Dirac operator (Section 6.8.3), and
- found why the Dunkl operator act the way they act (Section 6.8.5).

The original goal for this thesis was to make the correspondence "physically meaningful operators acting on conformal blocks" $\leftrightarrow$ "Dunkl operators" concrete by finding operators that correspond to the application of single Dunkl operators. However, as noted in the proof of Lemma 6.8.3, the only W-irreps we have access to when considering conformal blocks are $(+,+)$ and $(+,-)$, so in a sense there will always be some information lost when applying Dunkl operators to conformal blocks.

One topic that was also considered but ultimately not included in this thesis is the square of $\mathcal{D}$. In Theorem 4.3.5 we saw, purely algebraically, how $\mathcal{D}^2$ is related to the Casimir elements of $\mathfrak{g}$ and ∇. In particular, it leaves the $MA \times MA$-isotypes of $S \otimes V$ invariant, which for our choice of V gives it a matrix structure as follows:

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}$$

The diagonal elements are of the shape $\beta_{(+,\pm)}(S_{e_1}(\tilde{k})^2 + S_{e_2}(\tilde{k})^2)$ (like $ML^\pi(k)$, but in terms of Heckman–Dunkl operators, which might yield different results when π is nontrivial). Here, $\tilde{k}$ denotes a multiplicity vector adjusted according to the $MA \times MA$-isotype (due to the nontrivial action of A on elements of S, the Δ_1 is shifted), and the off-diagonal terms are 1st-order differential operators. Herein might lie the key to finding eigenfunctions of $\mathcal{D}$ (the *true* spinorial conformal blocks).

Other lines of inquiry as to the nature of the Dirac operator, expressed in terms of Dunkl operators, that were considered but not included here are

- perform a suitable (χ_1, χ_2-dependent) basis change to get rid of $\coth, \tanh$ (e.g. at the end of Section 6.8.3 and in Proposition 6.8.1),
- find connection d (e.g. the KZ connection) and metric such that $\mathcal{D} = \mathrm{d} + \mathrm{d}^\dagger$,
- shift operators,
- intertwiners (for W and for the affine Weyl group), and
- W-Spinors (a special case of which can be found at [CM09, section 4.5]).

Any of these, if successful, would provide a more intrinsically motivated expression for the expression on the Dunkl side. And finally, unlike for the Casimir element, where nontrivial M-representations require a qualitatively different computation than we saw here and is hard, the step from $V = \mathbb{C}$ to higher-dimensional representations is easier for the Dirac operator, and was in fact already carried out in Section 5.2. As mentioned at the end of Section 6.8.5, the consideration of more complicated representations of $MA \times MA$ might shed some more light on the correspondence of $MA \times MA$ representations and W-representations or on other "peripheral aspects" that go beyond just matching up the differential operators.

8. Popular Summary

Everyone who has ever cooked anything or made tea will be familiar with the following (very useful) phenomenon: when you heat water to its boiling point, it doesn't instantly boil off. Instead you have to keep supplying heat, and then small bubbles (of similar size!) of steam form. During the whole boiling process, the temperature stays constant at 100 degrees Celsius[1]. This extra heat that you have to supply to "force" a transition between different states of matter (a *phase transition*), is called *latent heat*.

There are, however, also phase transitions that have no latent heat. A magnet, for example, stops being magnetic if you heat it beyond a certain temperature. The states of "being magnetic" and "not being magnetic" turn out to be states of matter, too, and this loss of magnetism *is* a phase transition. But upon becoming non-magnetic, the magnet doesn't suddenly release big amounts of energy, and neither does it become non-magnetic only gradually. No, this is a change that happens without much "transition", more or less instantly, and without any latent heat. This is what we call a *2nd-order (or: continuous) phase transition*. Other, more visible (but probably less commonplace) examples of 2nd-order phase transitions include the transition between liquid and superfluid Helium, and the critical point of anything that has a liquid and a gas phase. The critical point is where the transition line between liquid and gas ends, the point beyond which there is no difference between gas and liquid anymore.

Unlike magnets, these last two examples have the advantage that we can work with see-through materials. In other words, we can *see* (and not only measure) what's going on. If we let such a phase transition happen slowly enough, we might be lucky and encounter *critical opalescence*: right at the phase transition the whole sample fogs up, becomes opaque (white and non-see-through), and then emerges in the other state of matter. One explanation of this phenomenon goes as follows: like with the boiling water, you have bubbles forming, and these bubbles interfere with the light coming through. However, unlike in boiling water, these bubbles come in all sizes, from micrometres to centimetres in diameter, because there is no surface tension. And they are also allowed to intersect and can form inside each other because there is no difference in density between the states of matter. Because of this chaos, we can't see individual bubbles, but just a general white mist, the critical opalescence.

This hints at a more general phenomenon: that right at this transition point, the system forgets about size or scale. The pattern of bubbles looks roughly the same at all sizes, and all physics happens equally on all scales. This means that things like density fluctuations, magnetisation fluctuations, and so on can be described using a theory called a *conformal field theory* (CFT). It turns out that CFTs can be described

[1] at sea level

using a relatively small set of data (the CFT data) that relates to physical observables such as critical exponents. Critical exponents characterise how quantities behave at and near the transition point: if they converge, if they diverge, and how they converge or diverge. The question, which CFT data really corresponds to a CFT, or physically speaking: which critical exponents could occur in nature, is a hard problem that involves solving an equation called the *crossing equation.* This problem is known as the *conformal bootstrap.*

The crossing equation is a functional equation, i.e. an equation whose solutions are *functions* that satisfy certain relations for *all possible inputs.* An example for a functional equation would be

$$f(x+y) = f(x)f(y) \qquad x, y \in \mathbb{R},$$

which is solved for example by the exponential function. In order to somewhat tame the crossing equation, we make an *Ansatz*: we guess that the solution is of the shape

$$\sum_{i \in I} a_i f_i,$$

i.e. a series of some form, for a collection $(f_i)_{i \in I}$ of “nice functions”, and a collection $(a_i)_{i \in I}$ of coefficients. If we plug this Ansatz into the crossing equation, we obtain infinitely many equations for the coefficients, which – with some luck and some smart ways of looking at it – we could attempt to solve. In order for us to be able to look in these smart ways, it is important that we know as much as possible about these *nice functions* f_i. They are called *conformal blocks*, and by some miracle they seem to be related to Heckman–Opdam hypergeometric functions. My thesis tries to advance the study of that miracle.

A. Miscellaneous Formalia

Let us now address some formal aspects that were mentioned in the text but not addressed properly, as well as provide some references.

A.1. Wightman Axioms

The Wightman axioms are a way of mathematically formalising what quantum fields (in Lorentzian QFT) are. Some free theories have been described in this framework but we are yet to find "nontrivial" interacting theories that can be thus described. [1] One starting-off point for the Wightman axioms, as well as other axiom systems, is [Sch08, chapter 8].

Let $\mathcal{H}$ be a Hilbert space, write $\mathscr{O}(\mathcal{H})$ for the set of densely defined linear operators on $\mathcal{H}$. For $A \in \mathscr{O}(\mathcal{H})$ write $D_A \subseteq \mathcal{H}$ for its domain. Let $\mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions on $\mathbb{R}^d$, and write P for the covering group of the Poincaré group.

Definition A.1.1. *A* field operator $\Psi \in \mathscr{F}\mathscr{O}(\mathcal{H})$ *is a function* $\mathcal{S}(\mathbb{R}^d) \to \mathscr{O}(\mathcal{H})$ *such that there is a dense linear subspace* $D \subseteq \mathcal{H}$ *such that*

(a) For all $f \in \mathcal{S}(\mathbb{R}^d)$ *we have* $D \subseteq D_{\Psi(f)}$*.*

(b) The map $\mathcal{S}(\mathbb{R}^d) \ni f \mapsto \Psi(f)|_D \in \mathrm{End}(D)$ *is linear.*

(c) For $w \in \mathcal{H}, v \in D$*, the assignment*

$$\mathcal{S}(\mathbb{R}^d) \ni f \mapsto \langle w, \Phi(f)v \rangle \in \mathbb{C}$$

is a tempered distribution.

Let (π, V) *be a finite-dimensional representation of a Lie group or algebra. Write* $\mathscr{T}\mathscr{F}\mathscr{O}(\mathcal{H}, V) := V \otimes \mathscr{F}\mathscr{O}(\mathcal{H})$ *for the space of* tensor field operators*, i.e. objects of the shape* $v^i\psi_i$ *for a basis* $(v^i)_{i\in I}$ *of* V *and field operators* ψ_i*.*

For $\psi \in \mathscr{T}\mathscr{F}\mathscr{O}(\mathcal{H}, V), \phi \in \mathscr{T}\mathscr{F}\mathscr{O}(\mathcal{H}, W$*, say*

$$\psi = v^i\psi_i, \qquad \phi = w^i\phi_i,$$

write

$$\psi(f) \otimes \phi(g) := v^i \otimes w^j \psi_i(f)\phi_j(g)$$

[1] In fact, mathematically rigorously describing a gapped Yang–Mills theory is one of the Millenium Prize Problems [Ins]

or as generalised functions

$$\psi(x) \otimes \phi(y) := v^i \otimes w^j \psi_i(x)\phi_j(y),$$

so the tensor product only acts on the "tensor" aspect of tensor field operators.

Similarly, for $w \in \mathcal{H}, u \in D, \psi \in \mathscr{T}\mathscr{F}\mathscr{O}(\mathcal{H}, V)$, say $v^i\phi_i$, write $\langle w, \phi v\rangle$ for the V-valued tempered distribution

$$\mathcal{S}(\mathbb{R}^d) \ni f \mapsto v^i\langle w, \phi_i(f)u\rangle.$$

For the special case when both v, w are the vacuum state, write $\langle\phi\rangle$.

A *Wightman quantum field theory* is a tuple $(\mathcal{H}, \Omega, U, \pi, (\psi_i)_{i\in I})$ of a separable (complex) Hilbert space $\mathcal{H}$, an element $\Omega \in \mathcal{H}$ with $\|\Omega\| = 1$, a unitary representation U of P on $\mathcal{H}$, a finite-dimensional representation of $\mathrm{Spin}(d-1, 1)$, and a tuple of field operators $\psi_i \in \mathscr{F}\mathscr{O}(\mathcal{H})$ such that the dense subspace D contained in all of their domains, contains Ω, subject to the following axioms:

Axiom 1 (Covariance)**.** *(a) For all $g \in P$ we have $U(p)D \subseteq D$ and $U(p)\Omega = \Omega$.*

(b) For all $f \in \mathcal{S}(\mathbb{R}^d)$ and $i \in I$ we have $\psi_i(f)D \subseteq D$.

(c) For $g = (\Lambda, a) \in P, f \in \mathcal{S}(\mathbb{R}^d)$ and $i \in I$ we have

$$U(g)\phi_i(f)U(g)^{-1} = \pi(\Lambda^{-1})^j{}_i\phi_j(g \cdot f)$$

where $(g \cdot f)(x) := f(\Lambda^{-1}(x - a))$.

If $(v^i)_{i\in I}$ is a basis of the representation space of π, realising

$$\pi(\Lambda)v^i = \pi(\Lambda)^i{}_j v_j,$$

then the Wightman QFT can also be phrased in terms of the tensor field operator (taking the group to be the covering group of the Lorentz group) $\psi := \psi_i v^i$ satisfies

$$U(g)\phi(f)U(g)^{-1} = \pi(g^{-1})\phi(g \cdot f).$$

Axiom 2 (Locality)**.** *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ have space-like separated supports, then we have*

$$[\psi_i(f), \psi_j(g)] = 0$$

on D.

Axiom 3 (Spectrum Condition)**.** *Let $P_0, \ldots, P_{d-1} \in \mathscr{O}(\mathcal{H})$ be the generators of the translation semigroups induced by U. Their joint spectrum is contained in the forward lightcone.*

A.2. Asymptotic Expansions

Asymptotic expansions are a tool that generalises Taylor and Laurent series, which is commonly used to describe functions f defined on 1-dimensional (complex) domains. They are a way of describing asymptotic behaviour around a point x in terms of functions of one's own choice (so-called *gauge functions*). Unlike Taylor and Laurent series we don't expect the asymptotic expansion of f to converge or, if it does converge, to equal f in any region. What matters is that the partial sums describe the asymptotic behaviour of f near x better and better. For more details, see [Mal]. We are describing a slightly generalised and specialised version of this that is able to describe functions in higher dimensions but uses a fixed choice of gauge functions.

Let $U \subseteq \mathbb{R}^d$ and $L \in \mathbb{R}^d$ a limit point of U.

Definition A.2.1. *A continuous function $f : U \setminus L \to \mathbb{R}$ is said to* have asymptotic expansion

$$\|x - L\|^{-n} \sum_{\alpha \geq 0} c_\alpha (x - L)^\alpha$$

around L *($n \in \mathbb{R}$, α is a multi-index) if for all $m \in \mathbb{N}$ we have*

$$\lim_{x \to L} \frac{\|x - L\|^n f - \sum_{|\alpha| \leq m} c_\alpha (x - L)^\alpha}{\|x - L\|^m} = 0,$$

i.e. if the difference between f and the m-th partial sum grows slower than $\|x - L\|^{m-n}$.

Example A.2.2. *Any multi-dimensional Taylor or Laurent expansion is an asymptotic expansion. In particular, for $d = 1$, $U = \mathbb{R}_{>0}$, and $L = 0$, the function $\exp(-t^{-1})$ has asymptotic expansion 0 around 0.*

Definition A.2.3. *A field operator ϕ has asymptotic expansion*

$$\|x - L\|^{-n} \sum_{\alpha \geq 0} c_\alpha (x - L)^\alpha$$

around L if for every $w \in \mathcal{H}$ and $v \in D$, the distribution $\langle w, \phi v \rangle$ has a regular domain of which L is a limit point, such that the regular function representing $\langle w, \phi v \rangle$ on that domain has asymptotic expansion

$$\|x - L\|^{-n} \sum_{\alpha \geq 0} \langle w, c_\alpha v \rangle (x - L)^\alpha.$$

A.3. Nuclear Spaces

When it comes to taking tensor products of vector spaces, things become sticky as you move to infinite dimensions. If V, W are Hilbert spaces, we can take their algebraic tensor product $V \otimes W$ and give it an inner product via

$$\langle v \otimes w, v' \otimes w' \rangle := \langle v, v' \rangle \langle w, w' \rangle$$

and then continuing it sesquilinearly. This then makes $V \otimes W$ into a pre-Hilbert space, which can be completed to form a Hilbert space – again called $V \otimes W$. This seems all well and good, but trouble arises as soon as we try to use "the" universal property of tensor products: representing bilinear maps. We could, for example, try to represent the bilinear map $ev : V^* \times V \to \mathbb{C}, (\lambda, v) \mapsto \lambda(v))$, which is wonderfully bounded to have operator norm 1, as element of $(V^*)^* \otimes V^* = V \otimes V^*$. If $(v_i)_{i \in \mathbb{N}}$ is an orthonormal basis of V, and $(\lambda_i)_{i \in \mathbb{N}}$ is its dual basis, then

$$\sum_{i=1}^{N} v_i \otimes \lambda_i$$

would represent ev on the space spanned by $v_1, \ldots, v_N$. Consequently, to represent ev on all of V, we'd have to take the limit $N \to \infty$, but alas! The N-th partial sum already has norm $\sqrt{N}$, so that series would never converge.

If we now widen our horizon slightly and consider Banach spaces, we can again follow the path of taking the algebraic tensor product, defining a suitable norm, and then completing with respect to that norm. In particular, our norm should satisfy:

Definition A.3.1. *Let V, V' be vector spaces, let N, N' be (semi)norms on V, V'. A map $p : V \otimes W \to \mathbb{R}$ (algebraic tensor product) is called* cross(semi)norm *if $p(v \otimes v') = N(v)N'(v')$ for $v \in V, v' \in V'$.*

Example A.3.2. *The* projective cross(semi)norm *of the seminorms N, N' is given by*

$$\pi(x) = \inf \left\{ \sum_{i=1}^{n} N(v_i) N'(v_i') \;\middle|\; x = \sum_{i=1}^{n} v_i \otimes v_i' \right\}.$$

The injective cross(semi)norm *of the seminorms N, N' is given by*

$$\epsilon(x) = \sup \left\{ |(\lambda \otimes \mu)(x)| \mid \lambda \in V^*, \mu \in V'^*, N^*(\lambda) = N'^*(\mu) = 1 \right\}$$

where N^* denotes the operator norm defined from N.

For two Banach spaces V, W, the projective and injective norm are generally different, and hence define different (completed) tensor products: the projective and injective tensor product. In some sense they can be seen as the "extremal" tensor products.

For Fréchet and other complete locally convex topological vector spaces, we can follow similar approaches and get a whole wealth of possible topological tensor products. For more details on various tensor products and their properties, consult [Trè67, part III]. In this setting, however, a class of locally convex Hausdorff topological vector spaces (LCHTVS) emerges that exceed all our expectations about their well-behavedness:

Definition A.3.3. *A LCHTVS V is called* nuclear *if for all LCHTVS W (equivalently, for all Banach spaces W), the (completed) projective and injective tensor products $V \otimes W$ are the same. More concretely, if the map*

$$V \otimes_\pi W \to V \otimes_\epsilon W$$

induced from the identity on the algebraic tensor products is an isomorphism of topological vector spaces.

In other words: among nuclear spaces there is really only one sensible notion of tensor product. It turns out, these spaces satisfy a host of wonderful "heredity" properties like

Proposition A.3.4. *Let V be a nuclear space.*

(a) The completion of V is nuclear.

(b) Any linear subspace of V is nuclear.

(c) Let $W \leq V$ be a closed subspace, then V/W is nuclear.

(d) Products of nuclear spaces are nuclear.

(e) Finite tensor products of nuclear spaces are nuclear.

Proof. See [Trè67, proposition 50.1]. □

But what we'd ultimately like is some interpretation for the tensor products between these spaces and, more importantly, some information of if they even exist (at least in the infinite-dimensional setting – all finite-dimensional vector spaces are nuclear)

Proposition A.3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be open. The spaces*

$$C^\infty(\Omega), C_c^\infty(\Omega), \mathcal{D}'(\Omega), \mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)$$

of smooth functions, test functions, distributions, Schwartz functions, and tempered distributions are nuclear.

Proof. This is the corollary after [Trè67, theorem 51.5]. □

And we have, as promised,

Proposition A.3.6.

$$\begin{aligned} \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{S}(\mathbb{R}^n) &\cong \mathcal{S}(\mathbb{R}^{m+n}) \\ \mathcal{S}'(\mathbb{R}^m) \otimes \mathcal{S}'(\mathbb{R}^n) &\cong \mathcal{S}'(\mathbb{R}^{m+n}) \\ C^\infty(X) \otimes C^\infty(Y) &\cong C^\infty(X \times Y) \\ \mathcal{D}'(X) \otimes \mathcal{D}'(Y) &\cong \mathcal{D}'(X \times Y) \end{aligned}$$

for $X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$ open.

Proof. This is [Trè67, theorem 51.6], as well as its corollary, and the Schwartz Kernel Theorem [Trè67, theorem 51.7]. □

These statements all hold for subsets of Euclidean space, but by covering a manifold M in patches $U_i \subseteq \mathbb{R}^n$ and imposing transition maps on the intersections (on the manifold), we can express $C^\infty(M), C_c^\infty(M), \mathcal{D}'(M)$ as linear subspaces of

$$\prod_i C^\infty(U_i), \qquad \prod_i C_c^\infty(U_i), \qquad \prod_i \mathcal{D}'(U_i),$$

showing that they are nuclear as well. A similar technique then also shows the statements about the interpretation of $\otimes$ on manifolds.

Nuclear spaces and the Schwartz Kernel Theorem are discussed in detail in [Trè67, sections 50& 51].

B. Some More Calculations

Most matrices in this appendix will be written as $(1, d, 1)$ block matrices.

B.1. Lie Algebra Elements

Recall that we defined $K^\mu := F^{0,\mu} - F^{d+1,\mu}$ and $P^\mu := -F^{0,\mu} - F^{d+1,\mu}$. This means that

$$\begin{aligned}(K^\mu)^\nu{}_\rho &= \eta^{0\nu}\delta^\mu_\rho - \eta^{\mu\nu}\delta^0_\rho - \eta^{d+1,\nu}\delta^\mu_\rho + \eta^{\mu\nu}\delta^{d+1}_\rho \\ &= \delta_{0,\nu}\delta_{\mu\rho} - \eta_{\mu\nu}\delta_{0,\rho} + \delta_{d+1,\nu}\delta_{\mu\rho} + \eta_{\mu\nu}\delta_{d+1,\rho},\end{aligned}$$

which corresponds to the matrix

$$\begin{pmatrix} 0 & e_\mu^T & 0 \\ -\eta e_\mu & 0 & \eta e_\mu \\ 0 & e_\mu^T & 0 \end{pmatrix}.$$

Similarly, P^μ corresponds to

$$\begin{pmatrix} 0 & -e_\mu^T & 0 \\ \eta e_\mu & 0 & \eta e_\mu \\ 0 & e_\mu^T & 0 \end{pmatrix}.$$

The contractions then look as follows:

$$\begin{aligned} b \cdot K = b_\mu K^\mu &= \begin{pmatrix} 0 & b_\bullet^T & 0 \\ -b^\bullet & 0 & b^\bullet \\ 0 & b_\bullet^T & 0 \end{pmatrix} \\ x \cdot P = x_\mu K^\mu &= \begin{pmatrix} 0 & -x_\bullet^T & 0 \\ x^\bullet & 0 & x^\bullet \\ 0 & x_\bullet^T & 0 \end{pmatrix}. \end{aligned}$$

For our algebra $\mathfrak{a}$ we have

$$D_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned}[K^\mu, P^\nu] &= -\Big[F^{0,\mu} - F^{d+1,\mu}, F^{0,\nu} + F^{d+1,\nu}\Big] \\ &= 2F^{\mu\nu} + 2\eta^{\mu\nu}F^{0,d+1} = 2F^{\mu\nu} + 2\eta^{\mu\nu}D_0\end{aligned}$$

B.2. Group Elements

In order to calculate $\exp(b \cdot K)$ and $\exp(x \cdot P)$ we calculate the squares of the contractions:

$$\begin{aligned}
(b \cdot K)^2 &= \begin{pmatrix} -b^2 & 0 & b^2 \\ 0 & 0 & 0 \\ -b^2 & 0 & b^2 \end{pmatrix} \\
(b \cdot K)^3 &= 0 \\
(x \cdot K)^2 &= \begin{pmatrix} -x^2 & 0 & -x^2 \\ 0 & 0 & 0 \\ x^2 & 0 & x^2 \end{pmatrix} \\
(x \cdot K)^3 &= 0
\end{aligned}$$

where $b^2 = \eta(b, b)$ and $x^2 = \eta(x, x)$. This shows that

$$\exp(b \cdot K) = \begin{pmatrix} 1 - \frac{b^2}{2} & b_\bullet^T & \frac{b^2}{2} \\ -b^\bullet & 1 & b^\bullet \\ -\frac{b^2}{2} & b_\bullet^T & 1 + \frac{b^2}{2} \end{pmatrix} \qquad \exp(x \cdot P) = \begin{pmatrix} 1 - \frac{x^2}{2} & -x_\bullet^T & -\frac{x^2}{2} \\ x^\bullet & 1 & x^\bullet \\ \frac{x^2}{2} & x_\bullet^T & 1 + \frac{x^2}{2} \end{pmatrix}.$$

Furthermore, we have

$$\begin{aligned}
w \exp(b \cdot K) w &= \begin{pmatrix} w_0 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & -w_0 \end{pmatrix} \begin{pmatrix} 1 - \frac{b^2}{2} & b_\bullet^T & \frac{b^2}{2} \\ -b^\bullet & 1 & b^\bullet \\ -\frac{b^2}{2} & b_\bullet^T & 1 + \frac{b^2}{2} \end{pmatrix} \begin{pmatrix} w_0 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & -w_0 \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{b^2}{2} & w_0 b_\bullet^T w & -\frac{b^2}{2} \\ -w_0 w b^\bullet & 1 & -w_0 w b^\bullet \\ \frac{b^2}{2} & -w_0 w b_\bullet^T & 1 + \frac{b^2}{2} \end{pmatrix} \\
&= \exp(-w_0 (wb) \cdot P).
\end{aligned}$$

Furthermore, we have

$$D_0^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that

$$\exp(\alpha D_0) = \begin{pmatrix} \cosh(\alpha) & 0 & \sinh(\alpha) \\ 0 & 1 & 0 \\ \sinh(\alpha) & 0 & \cosh(\alpha) \end{pmatrix}.$$

B.3. Group Action on Conformal Compactification

Let $x \in \mathbb{R}^{p,q}$ be non-isotropic. Recall the element w from the proof of Proposition 3.3.4, note that $A = \mathrm{diag}(a_0, \ldots, a_{d+1})$, where $a_0 = a_{d+1}$. Again, we also use A to denote the

$d \times d$ submatrix obtained by removing the first and last row and column. Then,

$$\begin{aligned}
w \cdot \iota(x) &= q\left(w\begin{pmatrix}1-\eta(x,x)\\ 2x\\ 1+\eta(x,x)\end{pmatrix}\right)\\
&= q\begin{pmatrix}w_0(1-\eta(x,x))\\ 2wx\\ -w_0(1+\eta(x,x))\end{pmatrix}\\
&= q\begin{pmatrix}\eta(x,x)-1\\ -2w_0wx\\ \eta(x,x)+1\end{pmatrix} = q\begin{pmatrix}1-\frac{1}{\eta(x,x)}\\ -\frac{2w_0wx}{\eta(x,x)}\\ 1+\frac{1}{\eta(x,x)}\end{pmatrix}\\
&= \iota\left(-\frac{w_0wx}{\eta(x,x)}\right) =: I(x)
\end{aligned}$$

B.4. $\overline{N}NMA$ Decomposition

Let $b, x \in \mathbb{R}^{p,q}$ as well as $\alpha \in \mathbb{R}$ and $m \in SO(p,q)$, then

$$\begin{aligned}
&\exp(x\cdot P)\exp(b\cdot K)\begin{pmatrix}\pm1 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & \pm1\end{pmatrix}\exp(\alpha D_0)\\
&=\begin{pmatrix}1-\frac{x^2}{2} & -x^T_\bullet & -\frac{x^2}{2}\\ x^\bullet & 1 & x^\bullet\\ \frac{x^2}{2} & x^T_\bullet & 1+\frac{x^2}{2}\end{pmatrix}\begin{pmatrix}1-\frac{b^2}{2} & b^T_\bullet & \frac{b^2}{2}\\ -b^\bullet & 1 & b^\bullet\\ -\frac{b^2}{2} & b^T_\bullet & 1+\frac{b^2}{2}\end{pmatrix}\begin{pmatrix}\pm1 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & \pm1\end{pmatrix}\begin{pmatrix}\cosh(\alpha) & 0 & \sinh(\alpha)\\ 0 & 1 & 0\\ \sinh(\alpha) & 0 & \cosh(\alpha)\end{pmatrix}\\
&=\begin{pmatrix}1+b_\mu x^\mu+\frac{-b^2-x^2+b^2x^2}{2} & -x^T_\bullet+(1-x^2)b^T_\bullet & -b_\mu x^\mu+\frac{b^2-x^2-b^2x^2}{2}\\ -b^\bullet+(1-b^2)x^\bullet & 1+2x^\bullet b^T_\bullet & b^\bullet+(1+b^2)x^\bullet\\ -b_\mu x^\mu+\frac{-b^2+x^2-b^2x^2}{2} & x^T_\bullet+(1+x^2)b^T_\bullet & 1+b_\mu x^\mu+\frac{b^2+x^2+b^2x^2}{2}\end{pmatrix}\begin{pmatrix}\pm\cosh(\alpha) & 0 & \pm\sinh(\alpha)\\ 0 & m & 0\\ \pm\sinh(\alpha) & 0 & \pm\cosh(\alpha)\end{pmatrix}\\
&=\begin{pmatrix}A & B^T_\bullet & C\\ D^\bullet & E & F^\bullet\\ G & H^T_\bullet & I\end{pmatrix}
\end{aligned}$$

where

$$
\begin{aligned}
A &= \pm\left(\frac{1-x^2}{2}\exp(\alpha) + \frac{1+2b_\mu x^\mu - b^2 + b^2x^2}{2}\exp(-\alpha)\right)\\
C &= \pm\left(\frac{1-x^2}{2}\exp(\alpha) - \frac{1+2b_\mu x^\mu - b^2 + b^2x^2}{2}\exp(-\alpha)\right)\\
G &= \pm\left(\frac{1+x^2}{2}\exp(\alpha) - \frac{1+2b_\mu x^\mu + b^2 + b^2x^2}{2}\exp(-\alpha)\right)\\
I &= \pm\left(\frac{1+x^2}{2}\exp(\alpha) + \frac{1+2b_\mu x^\mu + b^2 + b^2x^2}{2}\exp(-\alpha)\right)\\
B_\mu &= m^\nu{}_\mu\Big(-x_\nu + (1-x^2)b_\nu\Big)\\
D^\mu &= \pm\Big(\exp(\alpha)x^\mu - \exp(-\alpha)(b^\mu + b^2x^\mu)\Big)\\
F^\mu &= \pm\Big(\exp(\alpha)x^\mu + \exp(-\alpha)(b^\mu + b^2x^\mu)\Big)\\
H_\mu &= m^\nu{}_\mu\Big(x_\nu + (1+x^2)b_\nu\Big)\\
E &= m + 2x^\bullet b^T_\bullet m.
\end{aligned}
$$

From here we see that

$$
\begin{aligned}
\frac{A+C+G+I}{2} &= \pm\exp(\alpha)\\
\frac{A+C-G-I}{2} &= \mp x^2\exp(\alpha)\\
\frac{A-C+G-I}{2} &= \mp b^2\exp(-\alpha)\\
\frac{A-C-G+I}{2} &= \pm(1+2b_\mu x^\mu + b^2x^2)\exp(-\alpha)\\
\frac{D^\mu+F^\mu}{2} &= \pm\exp(\alpha)x^\mu\\
\frac{D^\mu-F^\mu}{2} &= \mp\exp(-\alpha)(b^\mu + b^2x^\mu)\\
\frac{B^T_\bullet + H^T_\bullet}{2} &= b^T_\bullet m,
\end{aligned}
$$

so that

$$\begin{aligned}
\alpha &= \log\left(\left|\frac{A+C+G+I}{2}\right|\right)\\
x^\mu &= \frac{D^\mu+F^\mu}{A+C+G+I}\\
b^\mu &= \frac{(F^\mu-D^\mu)(A+C+G+I)}{4}+\frac{(F^\mu+D^\mu)(A-C+G-I)}{4}\\
&= \frac{A+G}{2}F^\mu-\frac{C+I}{2}D^\mu\\
{m^\mu}_\nu &= {E^\mu}_\nu-\frac{(D^\mu+F^\mu)(B_\nu+H_\nu)}{A+C+G+I}.
\end{aligned}$$

Similarly, for $m\in SO(p,q), \alpha\in\mathbb{R}, b\in\mathbb{R}^{p,q}$ we have

$$\begin{aligned}
&w\exp(b\cdot K)\begin{pmatrix}\pm1 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & \pm1\end{pmatrix}\exp(\alpha D_0)\\
&=\begin{pmatrix}w_0 & 0 & 0\\ 0 & w & 0\\ 0 & 0 & -w_0\end{pmatrix}\begin{pmatrix}1-\frac{b^2}{2} & b_\bullet^T & \frac{b^2}{2}\\ -b^\bullet & 1 & b^\bullet\\ -\frac{b^2}{2} & b_\bullet^T & 1+\frac{b^2}{2}\end{pmatrix}\begin{pmatrix}\pm\cosh(\alpha) & 0 & \pm\sinh(\alpha)\\ 0 & m & 0\\ \pm\sinh(\alpha) & 0 & \pm\cosh(\alpha)\end{pmatrix}\\
&=\begin{pmatrix}w_0\left(1-\frac{b^2}{2}\right) & w_0 b_\bullet^T & w_0\frac{b^2}{2}\\ -wb^\bullet & w & wb^\bullet\\ w_0\frac{b^2}{2} & -w_0 b^\bullet & -w_0\left(1+\frac{b^2}{2}\right)\end{pmatrix}\begin{pmatrix}\pm\cosh(\alpha) & 0 & \pm\sinh(\alpha)\\ 0 & m & 0\\ \pm\sinh(\alpha) & 0 & \pm\cosh(\alpha)\end{pmatrix}\\
&=\begin{pmatrix}\pm\frac{w_0}{2}\exp(\alpha)\pm w_0\frac{1-b^2}{2}\exp(-\alpha) & w_0 b_\bullet^T m & \pm\frac{w_0}{2}\exp(\alpha)\mp w_0\frac{1-b^2}{2}\exp(-\alpha)\\ \mp\exp(-\alpha)wb^\bullet & wm & \pm\exp(-\alpha)wb^\bullet\\ \mp\frac{w_0}{2}\exp(\alpha)\pm w_0\frac{1+b^2}{2}\exp(-\alpha) & -w_0 b^\bullet m & \mp\frac{w_0}{2}\exp(\alpha)\mp w_0\frac{1+b^2}{2}\exp(-\alpha)\end{pmatrix}\\
&=\begin{pmatrix}A & B_\bullet^T & C\\ D^\bullet & E & F^\bullet\\ G & H_\bullet^T & I\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \log\left(\left|\frac{A+C-G-I}{2}\right|\right)\\
b^\mu &= {w^\mu}_\nu\frac{(H^\nu-F^\nu)(A+C-G-I)}{4}\\
{m^\mu}_\nu &= {w^\mu}_\rho {E^\rho}_\nu.
\end{aligned}$$

Proposition B.4.1.

$$\exp(b\cdot K)\exp(x\cdot P)=\exp(x'\cdot P)\exp(b'\cdot K)m\exp(\alpha D_0)$$

where

$$\begin{aligned}
\alpha &= \log\Big(1 + 2b_\mu x^\mu + b^2 x^2\Big) \\
x'^\mu &= \frac{x^\mu + x^2 b^\mu}{1 + 2b_\nu x^\nu + b^2 x^2} \\
b'^\mu &= (1 + x^\nu b_\nu) b^\mu + b_\nu (x^\nu b^\mu - x^\mu b^\nu) \\
m^\mu{}_\nu &= \delta^\mu_\nu + 2b^\mu x_\nu - 2\frac{(x^\mu + x^2 b^\mu)(b_\nu + b^2 x_\nu)}{1 + 2b_\rho x^\rho + b^2 x^2}
\end{aligned}$$

if $0 \neq 1 + 2b_\mu x^\mu + b^2 x^2$, *and* $w \exp(b' \cdot K) m \exp(\alpha D_0)$ *with*

$$\begin{aligned}
\alpha &= \log\Big(\Big|x^2\Big|\Big) \\
b'^\mu &= -x^2 m^\mu{}_\nu b^\nu \\
m^\mu{}_\nu &= w^\mu{}_\rho (\delta^\rho_\nu + 2b^\rho x_\nu)
\end{aligned}$$

otherwise.

Proof.

$$\begin{aligned}
\exp(b \cdot K)\exp(x \cdot P) &= \begin{pmatrix} 1 - \frac{b^2}{2} & b^T_\bullet & \frac{b^2}{2} \\ -b^\bullet & 1 & b^\bullet \\ -\frac{b^2}{2} & b^T_\bullet & 1 + \frac{b^2}{2} \end{pmatrix} \begin{pmatrix} 1 - \frac{x^2}{2} & -x^T_\bullet & -\frac{x^2}{2} \\ x^\bullet & 1 & x^\bullet \\ \frac{x^2}{2} & x^T_\bullet & 1 + \frac{x^2}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 + b_\mu x^\mu + \frac{-b^2 - x^2 + b^2 x^2}{2} & b^T_\bullet - (1 - b^2) x^T_\bullet & b_\mu x^\mu + \frac{b^2 - x^2 + b^2 x^2}{2} \\ x^\bullet - (1 - x^2) b^\bullet & 1 + 2b^\bullet x^T_\bullet & x^\bullet + (1 + x^2) b^\bullet \\ b_\mu x^\mu + \frac{-b^2 + x^2 + b^2 x^2}{2} & b^T_\bullet + (1 + b^2) x^T_\bullet & 1 + b_\mu x^\mu + \frac{b^2 + x^2 + b^2 x^2}{2} \end{pmatrix} \\
&= \begin{pmatrix} A & B^T_\bullet & C \\ D^\bullet & E & F^\bullet \\ G & H^T_\bullet & I \end{pmatrix},
\end{aligned}$$

so that

$$\begin{aligned}
\alpha &= \log\left(\left|\frac{A + C + G + I}{2}\right|\right) = \log\Big(1 + 2b_\mu x^\mu + b^2 x^2\Big) \\
x'^\mu &= \frac{D^\mu + F^\mu}{A + C + G + I} = \frac{x^\mu + x^2 b^\mu}{1 + 2b_\nu x^\nu + b^2 x^2} \\
b'^\mu &= \frac{A + G}{2} F^\mu - \frac{C + I}{2} D^\mu \\
&= (1 + 2b_\nu x^\nu + b^2 x^2) b^\mu - b^2 (x^\mu + x^2 b^\mu) \\
&= b^\mu + x^\nu b_\nu b^\mu + b_\nu (x^\nu b^\mu - x^\mu b^\nu) \\
m &= 1 + 2b^\bullet x^T_\bullet - 2\frac{(x^\bullet + x^2 b^\bullet)(b^T_\bullet + b^2 x^T_\bullet)}{1 + 2b_\mu x^\mu + b^2 x^2}
\end{aligned}$$

for an $\overline{N}NMA$ decomposition.

For the case that $b \in \mathcal{O}(t)$ and we're truncating after order t, we have

$$\begin{aligned}
\alpha &= 2b_\mu x^\mu \\
x'^\mu &= \frac{x^\mu}{1+2b_\nu x^\nu} + x^2 b^\mu = x^\mu(1-2b_\nu x^\nu) + x^2 b^\mu \\
b'^\mu &= b^\mu \\
m^\mu{}_\nu &= 1 + 2b^\mu x_\nu - 2x^\mu b_\nu.
\end{aligned}$$

If we have a $wMAN$ decomposition instead, the parameters are as follows:

$$\begin{aligned}
\alpha &= \log\left(\left|\frac{A+C-G-I}{2}\right|\right) = \log\left(\left|x^2\right|\right) \\
b'^\mu &= w^\mu{}_\nu \frac{(H^\nu - F^\nu)(A+C-G-I)}{4} = -x^2 w^\mu{}_\nu b^\nu \\
m^\mu{}_\nu &= w^\mu{}_\rho(\delta^\rho_\nu + 2b^\rho x_\mu).
\end{aligned}$$

□

Proposition B.4.2.

$$w\exp(x\cdot P) = \exp(x'\cdot P)\exp(b'\cdot K) m \exp(\alpha D_0)$$

where $x' = I(x)$.

Proof.

$$\begin{aligned}
w\exp(x\cdot P) &= w\begin{pmatrix} 1-\frac{x^2}{2} & -x_\bullet^T & -\frac{x^2}{2} \\ x^\bullet & 1 & x^\bullet \\ \frac{x^2}{2} & x_\bullet^T & 1+\frac{x^2}{2} \end{pmatrix} \\
&= \begin{pmatrix} w_0\left(1-\frac{x^2}{2}\right) & -w_0 x_\bullet^T & -w_0\frac{x^2}{2} \\ wx^\bullet & w & wx^\bullet \\ -w_0\frac{x^2}{2} & -w_0 x_\bullet^T & -w_0\left(1+\frac{x^2}{2}\right) \end{pmatrix},
\end{aligned}$$

so that

$$\begin{aligned}
\alpha &= \log\left(\left\|x^2\right\|\right) \\
x'^\mu &= -\frac{w_0 w^\mu{}_\nu x^\nu}{x^2} \\
b'^\mu &= w_0 w^\mu{}_\nu x^\nu \\
m^\mu{}_\nu &= w^\mu{}_\rho\left(\delta^\rho_\nu - \frac{2x^\rho x_\nu}{x^2}\right)
\end{aligned}$$

□

B.5. $MA\overline{N}w\overline{N}MA$ Decomposition

Let $x, y \in \mathbb{R}^{p,q}, \alpha, \beta \in \mathbb{R}, m, m' \in SO(p,q)$, then let's calculate

$$m \exp(\alpha D_0) \exp(x \cdot P) w \exp(y \cdot P) m' \exp(\beta D_0).$$

First recall that the elements of MA are

$$\begin{pmatrix} \sigma(m)\cosh(\alpha) & 0 & \sigma(m)\sinh(\alpha) \\ 0 & m & 0 \\ \sigma(m)\sinh(\alpha) & 0 & \sigma(m)\cosh(\alpha) \end{pmatrix}, \qquad (m, \alpha \leftrightarrow m', \beta),$$

where $\sigma(m) \in \{\pm 1\}$ such that

$$\begin{pmatrix} \sigma(m) & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \sigma(m) \end{pmatrix} \in G.$$

Furthermore, for $\exp(x \cdot P) w \exp(y \cdot P)$ we get

$$\begin{pmatrix} 1 - \frac{x^2}{2} & -x_\bullet^T & -\frac{x^2}{2} \\ x^\bullet & 1 & x^\bullet \\ \frac{x^2}{2} & x_\bullet^T & 1 + \frac{x^2}{2} \end{pmatrix} \begin{pmatrix} w_0 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & -w_0 \end{pmatrix} \begin{pmatrix} 1 - \frac{y^2}{2} & -y_\bullet^T & -\frac{y^2}{2} \\ y^\bullet & 1 & y^\bullet \\ \frac{y^2}{2} & y_\bullet^T & 1 + \frac{y^2}{2} \end{pmatrix}$$
$$= \begin{pmatrix} w_0 - w_{\mu\nu} x^\mu y^\nu + w_0 \frac{-x^2 - y^2 + x^2 y^2}{2} & -w_0(1 - x^2) y_\bullet^T - x_\bullet^T w & -w_{\mu\nu} x^\mu y^\nu + w_0 \frac{x^2 - y^2 + x^2 y^2}{2} \\ w_0(1 - y^2) x^\bullet + w y^\bullet & w - 2 w_0 x^\bullet y_\bullet^T & -w_0(1 + y^2) x^\bullet + w y^\bullet \\ w_{\mu\nu} x^\mu y^\nu + w_0 \frac{x^2 - y^2 - x^2 y^2}{2} & -w_0(1 + x^2) y_\bullet^T + x_\bullet^T w & -w_0 + w_{\mu\nu} x^\mu y^\nu + w_0 \frac{-x^2 - y^2 - x^2 y^2}{2} \end{pmatrix}.$$

Thus, multiplying it all together, we get

$$\begin{pmatrix} A & * & C \\ * & * & * \\ G & * & I \end{pmatrix}$$

with

$$
\begin{aligned}
A &= \sigma(m)\sigma(m')\Big(-\frac{w_0 y^2}{2}\exp(\alpha+\beta) + \frac{w_0}{2}\exp(\alpha-\beta) + \frac{w_0 + w_0 x^2 y^2 - 2w_{\mu\nu}x^\mu y^\nu}{2}\exp(-\alpha+\beta) \\
&\qquad - \frac{w_0 x^2}{2}\exp(-\alpha-\beta)\Big) \\
C &= \sigma(m)\sigma(m')\Big(-\frac{w_0 y^2}{2}\exp(\alpha+\beta) - \frac{w_0}{2}\exp(\alpha-\beta) + \frac{w_0 + w_0 x^2 y^2 - 2w_{\mu\nu}x^\mu y^\nu}{2}\exp(-\alpha+\beta) \\
&\qquad + \frac{w_0 x^2}{2}\exp(-\alpha-\beta)\Big) \\
G &= \sigma(m)\sigma(m')\Big(-\frac{w_0 y^2}{2}\exp(\alpha+\beta) + \frac{w_0}{2}\exp(\alpha-\beta) - \frac{w_0 + w_0 x^2 y^2 - 2w_{\mu\nu}x^\mu y^\nu}{2}\exp(-\alpha+\beta) \\
&\qquad + \frac{w_0 x^2}{2}\exp(-\alpha-\beta)\Big) \\
I &= \sigma(m)\sigma(m')\Big(-\frac{w_0 y^2}{2}\exp(\alpha+\beta) - \frac{w_0}{2}\exp(\alpha-\beta) - \frac{w_0 + w_0 x^2 y^2 - 2w_{\mu\nu}x^\mu y^\nu}{2}\exp(-\alpha+\beta) \\
&\qquad - \frac{w_0 x^2}{2}\exp(-\alpha-\beta)\Big).
\end{aligned}
$$

This shows that

$$
\begin{aligned}
\frac{A+C+G+I}{2\sigma(m)\sigma(m')w_0} &= y^2\exp(\alpha+\beta) \\
\frac{A+C-G-I}{2\sigma(m)\sigma(m')w_0} &= (1 - 2w_0 w_{\mu\nu}x^\mu y^\nu + x^2 y^2)\exp(-\alpha+\beta) \\
\frac{A-C+G-I}{2\sigma(m)\sigma(m')w_0} &= \exp(\alpha-\beta) \\
\frac{A-C-G+I}{2\sigma(m)\sigma(m')w_0} &= -x^2\exp(-\alpha-\beta).
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
\frac{(C+G)^2-(A+I)^2}{4} &= x^2 y^2 \\
\frac{(A-I)^2-(C-G)^2}{4} &= 1 - 2w_0 w_{\mu\nu}x^\mu y^\nu + x^2 y^2 \\
\log\left(\left|\frac{x^2(A-C+G-I)}{A-C-G+I}\right|\right) &= 2\alpha \\
\log\left(\left|\frac{4x^2}{(A-C)^2-(G-I)^2}\right|\right) &= 2\beta.
\end{aligned}
$$

If we assume that $x^2 = \pm 1$, then

$$\begin{aligned}
\pm\frac{(C+G)^2-(A+I)^2}{4} &= y^2 \\
\pm\frac{(A-I)^2-(C-G)^2}{4} &= (y \mp w_0 w x)^2 \\
\log\left(\left|\frac{(A-C+G-I)}{A-C-G+I}\right|\right) &= 2\alpha \\
\log\left(\left|\frac{4}{(A-C)^2-(G-I)^2}\right|\right) &= 2\beta.
\end{aligned}$$

For $(x,y) \in Y_{\sigma\tau}$ from Proposition 3.3.9 we see that the scalars y^2 and $(y \mp w_0 w x)^2$ are precisely those that occur in (3.3). In conclusion, for

$$U \ni x = \begin{pmatrix} A & * & C \\ * & * & * \\ G & * & I \end{pmatrix}$$

we have

$$u(x) = \frac{4}{(A-I)^2-(C-G)^2}, \qquad v(x) = \frac{(C+G)^2-(A+I)^2}{(A-I)^2-(C-G)^2}$$

and $x \in M \exp(\alpha D_0) Y_{\sigma\tau} \exp(\beta D_0) M$ for

$$\alpha = \frac{1}{2}\log\left(\left|\frac{(A-C+G-I)}{A-C-G+I}\right|\right), \qquad \beta = \frac{1}{2}\log\left(\left|\frac{4}{(A-C)^2-(G-I)^2}\right|\right).$$

B.6. Embedding α_Y

We begin by evaluating our invariant bilinear form B. For that first note that

$$\begin{aligned}
\operatorname{tr}(E^{\mu\nu}E^{\rho\sigma}) &= E^{\mu\nu\alpha}{}_{\beta} E^{\rho\sigma\beta}{}_{\alpha} \\
&= \eta^{\mu\alpha}\delta^{\nu}_{\beta}\eta^{\rho\beta}\delta^{\sigma}_{\alpha} \\
&= \eta^{\mu\sigma}\eta^{\nu\rho}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
B(F^{\mu\nu}, F^{\rho\sigma}) &= \operatorname{tr}(E^{\mu\nu}E^{\rho\sigma} - E^{\nu\mu}E^{\rho\sigma} - E^{\mu\nu}E^{\sigma\rho} + E^{\nu\mu}E^{\sigma\rho}) \\
&= 2\eta^{\mu\sigma}\eta^{\nu\rho} - 2\eta^{\mu\rho}\eta^{\nu\sigma}.
\end{aligned}$$

This shows that $\left(-\frac{1}{2}F_{\mu\nu}\right)_{\mu<\nu}$ is the dual basis of $(F^{\mu\nu})_{\mu<\nu}$ of $\mathfrak{m}$. From this, we can conclude that

$$B(D_0, D_0) = B(F^{0,d+1}, F^{0,d+1}) = -2\eta^{00}\eta^{d+1,d+1} = 2$$

and that

$$\begin{aligned} B(P^\mu, K^\nu) &= B(-F^{0,\mu} - F^{d+1,\mu}, F^{0,\nu} - F^{d+1,\nu}) \\ &= 2\eta^{00}\eta^{\mu\nu} - 2\eta^{d+1,d+1}\eta^{\mu\nu} \\ &= 4\eta^{\mu\nu}. \end{aligned}$$

This shows that the bases

$$\frac{1}{4}P_1, \ldots, \frac{1}{4}P_d, \frac{1}{4}K_1, \ldots, \frac{1}{4}K_d$$

and $K^1, \ldots, K^d, P^1, \ldots, P^d$ of $Y = \mathfrak{n} \oplus \overline{\mathfrak{n}}$ are dual.

As a consequence, the quadratic Casimir elements of $\mathfrak{g}$ and $\mathfrak{r}$ can be written as

$$\begin{aligned} \Omega_{\mathfrak{g}} &= -\frac{1}{2}\sum_{1 \le \mu < \nu \le d} F_{\mu\nu}F^{\mu\nu} + \frac{D_0^2}{2} + \frac{1}{4}P_\mu K^\mu + \frac{1}{4}K^\mu P_\mu \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_0^2 + \frac{1}{4}\{P^\mu, K_\mu\} \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_0(D_0 + d) + \frac{1}{2}P^\mu K_\mu \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_0(D_0 - d) + \frac{1}{2}K^\mu P_\mu \\ \Omega_{\mathfrak{r}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_0^2. \end{aligned}$$

Furthermore, the map α_Y from (4.1) can be written as

$$\begin{aligned} \alpha_Y(\xi) = \frac{1}{64}\Big(&B(\xi, [P_\mu, P_\nu])K^\mu K^\nu + B(\xi, [P_\mu, K_\nu])K^\mu P^\nu \\ &+ B(\xi, [K_\nu, P_\mu])P^\mu K^\nu + B(\xi, [K_\mu, K_\nu])P^\mu P^\nu\Big). \end{aligned}$$

Using that the P's and K's commute among each other and that $[K_\mu, P_\nu] = 2F_{\mu\nu} + 2\eta_{\mu\nu}D_0$, we obtain

$$\alpha_Y(\xi) = \frac{1}{32}(B(\xi, F_{\mu\nu} - \eta_{\mu\nu}D_0)K^\mu P^\nu + B(\xi, F_{\mu\nu} + \eta_{\mu\nu}D_0)P^\mu K^\nu).$$

For the case $\xi = F^{\mu\nu}$ we obtain

$$\begin{aligned} \alpha_Y(F^{\mu\nu}) &= \frac{1}{16}(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma})(K^\rho P^\sigma - K^\sigma P^\rho) \\ &= \frac{1}{8}(K^\nu P^\mu - K^\mu P^\nu) \end{aligned}$$

where we used that the coefficient in front of terms with $\sigma = \rho$ is 0, so K^σ, P^ρ anticommute.

For the case $\xi = D_0$ we obtain

$$\begin{aligned}
\alpha_Y(D_0) &= \frac{1}{16}(-K^\mu P_\mu + P_\mu K^\mu) \\
&= \frac{1}{16}[P^\mu, K_\mu] \\
&= \frac{1}{8}(P^\mu K_\mu + 4d) = -\frac{1}{8}(K^\mu P_\mu + 4d).
\end{aligned}$$

Finally, we can now use this to calculate $\alpha_Y(\Omega_\mathfrak{r})$:

$$\begin{aligned}
\frac{1}{4}\alpha_Y(F_{\mu\nu}F^{\mu\nu}) &= \frac{1}{256}\sum_{\mu\neq\nu}(K_\nu P_\mu - K_\mu P_\nu)(K^\nu P^\mu - K^\mu P^\nu) \\
&= \frac{1}{128}\sum_{\mu\neq\nu}(K_\nu P_\mu K^\nu P^\mu - K_\mu P_\nu K^\nu P^\mu) \\
&= \frac{1}{128}\sum_{\mu\neq\nu}(-K_\nu K^\nu P_\mu P^\mu + 8\delta^\nu_\nu K^\mu P_\mu + K_\mu K^\nu P_\nu P^\mu) \\
&= \frac{d-1}{16}K^\mu P_\mu - \frac{1}{128}K^\mu K^\nu P_\mu P_\nu \\
\frac{1}{2}\alpha_Y(D_0)^2 &= \frac{1}{128}(4d + K^\mu P_\mu)(4d + K^\nu P_\nu) \\
&= \frac{d^2}{8} + \frac{d}{16}K^\mu P_\mu + \frac{1}{128}K^\mu P_\mu K^\nu P_\nu \\
&= \frac{d^2}{8} + \frac{d}{16}K^\mu P_\mu + \frac{1}{128}(-8K^\mu P_\nu - K^\mu K^\nu P_\mu P_\nu) \\
&= \frac{d^2}{8} + \frac{d-1}{16}K^\mu P_\mu - \frac{1}{128}K^\mu K^\nu P_\mu P_\nu.
\end{aligned}$$

And thus,

$$\alpha_Y(\Omega_\mathfrak{r}) = \frac{d^2}{8}.$$

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